Asymptotic Methods and Perturbation Theory

Course contents:

Mathematical equations arising from engineering and physical problems are intractable analytically. Engineers and scientists are therefore forced to make approximations. Perturbation and asymptotic methods are among the most important tools available to engineers and scientist for obtaining rational and reliable approximations. Taking advantages of the relative magnitude of the different controlling parameters and/or the disparate scales, a complicated problem is replaced by a set of simpler problems that can be either solved analytically or, if a numerical solution is required, can be solved a relatively simple strategy. This procedure permits the construction of reasonably accurate solutions with deep physical insight. An awareness of the structure of the solution obtained by perturbation methods is often helpful even when a direct numerical simulation of the full problem is adopted. Perturbation and numerical methods, therefore, complement one another.
The goal in this course is to convey the main ideas of perturbation theory by illustrating the approach on examples taken from the various engineering disciplines and the physical sciences. The discussion will cover regular perturbation problems, singular perturbation problems of secular-type and layer-type, linear and nonlinear stability and bifurcation theory. The methods of multiscale, Krylov-Bogoliubov averaging, strained coordinates, matched asymptotic expansions, homogenization, WKB, Laplace's method for integrals and the method of steepest descent will be introduced, along with the notations of ordering, asymptotic expansions, limit process expansions, uniform expansions and distinguished limits.

Subjects covered by the lectures are

- Dimensional Analysis. Element of mathematical modelling
- Expansion of functions and mathematical methods
- Mathematical methods of perturbations
- Regular and singular perturbations
- Wave-impact processes
- Pade approximations
- Averaging of ribbed plates
- Chaos foresight
- Continuous approximation of discontinuous systems
- Nonlinear dynamics of a swinging oscillator
- The Homotopy Analysis Method

Learning outcomes of the course :

Through a deep understanding of the theory and the realization of a project, the student will be able to apply asymptotical methods to solve mechanical problems. In particular:

- He will have a deep understanding of perturbation theories and asymptotical methods and will be able to summary, compare and explain them.
- He will have a deep understanding of the resolution methods of oscillation problems, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to apply the resolution methods to classical problems and new problems.
- He will be able to analyze and to evaluate (justify and criticise) these methods.
- He will be able to analyze new problems.

Prerequisites and co-requisites/ Recommended optional programme components :

Basic knowledge in

- Ordinary Differential Equations
- Partial Differential Equations
- Elasticity Theory
- Fluid Mechanics

Planned learning activities and teaching methods :

Exercises with professor assistance and personal project.

Mode of delivery (face-to-face ; distance-learning) :

Face-to-Face

Required readings :

Basic literature

1. Nayfeh A.H. Introduction to Perturbation Techniques. New York, Chchester, Brisbane, Toronto, Singapor: Wlley&Son, 1993. 519 p.
2. Holmes M.N. Introduction to Perturbation Methods. Springer-Verlag, 1995.
3. Van Dyke M. Perturbation Methods in Fluid Dynamics. Parabolic Press.
4. Kevorkian J., Cole J.D. Multiscale and Singular Perturbation Methods. Springer-Verlag,
5. Andrianov I.V., Awrejcewicz J., Manevitch L.I. Asymptotical Mechanics of Thin-Walled Structures: a Handbook. – Berlin, Heidelberg: Springer-Verlag, 2004. – 535 p.
6. Landa P.S. Regular and Chaotic Oscillations. – Berlin, New York: Springer, 2001. – 395 p.
7. Manevitch L.I., Andrianov I.V., Oshmyan V.G. Mechanics of Periodically Heterogeneous Structures. – Berlin: Springer, 2002. – 264 p.
8. Liao S. Beyond Perturbation. Introduction to the homotopy analysis method. Boca Raton: Charman&Hall/CRC, 2004.
9. A.H. Nayfeh, D.T. Mook, Nonlinear oscillations. New York: John Wiley and Sons; 1995.
10. J. Awrejcewicz, V.A. Krysko, Introduction to Asymptotic Methods. Boca Raton, New York: Chapman and Hall/CRC Press; 2006.
11. I.V. Andrianov, J. Awrejcewicz, L.I. Manevitch, Asymptotical Mechanics of Thin Walled Structures. A Handbook. Berlin: Springer-Verlag; 2004.
12. J. Awrejcewicz, Bifurcation and Chaos in Coupled Oscillations. Singapore: World Scientific, 1991.
13. I.V. Andrianov, J. Awrejcewicz, New trends in asymptotic approaches: summation and interpolation methods, Applied Mechanics Reviews. 54(1)(2001) 69-92.
14. G.A. Baker, P. Graves-Morris, Pade Approximants. London, Amsterdam, Don Mills, Ontario, Sydney, Tokio: Addison-Wesley Publishing Com; 1981.
15. I.V. Andrianov, L.I. Manevich, Asymptotology: Ideas, Methods, and Applications. Dordrecht, Boston, London: Kluwer Academic Publisher; 2002.
16. Nonlinearity, Bifurcation and Chaos - Theory and Application. Edited by J. Awrejcewicz and P. Hagedorn. Rijecka: InTech; 2012.
17. Kudryashov N.A. Methods of Nonlinear Mathematical Physics. Dolgoprudnyi: Publishing House "Intellect"; 2010.
18. Li Recho N. Methodes asymptotiques en mecanique de la rupture. Paris: Hermes Science Publications; 2002.
19. Stepanova L.V. Mathematical methods of fracture mechanics. Moscow: Fizmatlit; 2009. 336 p.
20. A.D. Polyanin, V.F. Zaitsev, A.I. Zhurov, Methods of Solving Nonlinear Equations of Mathematical Physics and Mechanics. Moscow: Fizmatlit; 2005.

21. A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations. Boca raton: Chapman and Hall/CRC Press; 2004.
22. Bender C.M., Orszag S.A. Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory (v. 1). Springer, 1999.
23. W. E. Boyce and R. C. DiPrima, Elementary Di_erential Equations and Boundary Value Problems, 9th ed., John Wiley and Sons, Hoboken, New Jersey, 2009.
24. Paulsen W. Asymptotic Analysis and Perturbation Theory. Boca Raton, London, New York: CRC Press, 20014. 546 p.
25. Ablowitz M.J. Nonlinear Dispersive Waves. Asymptotic Analysis and Solitons. Cambridge University Press, 2011. 364 p.

Assessment methods and criteria :

Evaluation is based on the realization of a project related to the use / development of asymptotic methods and perturbation theory specific to solid mechanics problems and on an examination.
The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked.
Participation to the examination and achievement of the project are mandatory.

Teaching Method: Class participation is mandatory. Everyone is expected to participate in discussions relating to reading materials, homework, exams and lectures.

Guaranteed Recipe for Success:
1) Take notes during lecture and sections.
2) After each lecture but before the next lecture review your notes. Identify the parts you do not understand.
3) Come to each lecture and discussion section with specific questions.
4) Keep up with the reading so that you have some familiarity with each topic prior to hearing about it in the lecture.
5) Find at least one "partner" in the class with whom you can meet at least once or twice a week to discuss materials from the lectures, the reading assignments and the homework.
6) Take the homework assignment seriously. Do not try to do the whole assignment the night before it is due. Some version of the homework questions will appear on the exams.

Asymptotic Methods and Perturbation Theory
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1. Introduction

1.1. Dimensional Analysis 1.2. Expansions 1.3. Gauge Functions 1.4. Order Symbols 1.5. Asymptotic Series 1.6. Asymptotic Expansions and Sequences 1.7. Convergent Versus Asymptotic Series 1.8. Elementary Operations on Asymptotic Expansions

2. Algebraic Equations

2.1. Quadratic Equations 2.2. Cubic Equations 2.3. Higher-Order Equations 2.4. Transcendental Equations

3. Integrals

3.1. Expansion of Integrands 3.2. Integration by Parts 3.4. Laplace's Method 3.5. The methd of Steepest Descent

4. The Duffing Equation

4.1. The Straightforward Expansion 4.2. Exact Solution 4.3. The Lindstedt-Poincare Technique 4.4. The Method of Renormalization 4.5.The Method of Multiple Scales 4.6. Variation of Parameters 4.7. The Method of Averaging

5. The Linear Damped Oscillator

5.1. The Straightforward Expansion 5.2. Exact Solution 5.3. The Lindstedt-Poincare Technique 5.4. The Method of Multiple Scales 5.5. The Method of Averaging

6. Self-Exited Oscillators

6.1. The Straightforward Expansion 6.2. The Method of Renormalization 6.3. The Method of Multiple Scales 6.4. The Method of Averaging

7. Systems with Quadratic and Cubic Nonlinearities

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8. General Weakly Nonlinear Systems

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9. Boundary-Layer Problems

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10. Differential Equations with a Large Parameter

10.1. The WKB Approximation 10.2. The Liouville-Green Transformation 10.3. Eigenvalue problems 10.4. Equations with Slowly Varying Coefficients 10.5. Turning-Points Problems 10.6. The Langer Transformation 10.7. Eigenvalue Problems with Turning Points

11. Solvability Conditions

11.1. Algebraic Equations 11.2. Nonlinear Vibrations of Two-Degree-of Freedom Gyroscopic Systems 11.3. Parametrically Excites Gyroscopic System 11.4. Second-Order Differential Systems 11.5. General Boundary Conditions 11.6. A Simple Eigenvalue Problem 11.7. Degenerate Eigenvalue Problems 11.8. Acoustic Waves in Duct with Sinisoidal Walls 11.9. Vibrations of Nearly Circular Membranes 11.10. A Fourth-Order Differential System 11.11. General Fourth-Order Eigenvalue Problem 11.12. A Fourth –Order Eigenvalue Problem 11.13. A Differential System of Equations 11.14. General Differential Systems of First-Order Equations 11.15. Differential Systems with Interfacial Boundary Conditions 11.16. Integral Equations 11.17. Partial-Differential Equations 12. Pade Approximations 12.1. Determination and characteristics of Pade approximations 12.2. Application of Pade approximations 12.3. Simple examples 12.4. Supersonic Flow round a thin cone in circumsonic regime 12.5. Damping of the ball-shaped waves of pressure in a free space and in a tube 12.6 Analysis of the "blow-up" phenomenon 12.7. Homoclinic orbits 12.8. Vibrations of nonlinear system with nonlinearity close to sign(x)

Basic Ideas of The Homotopy Analysis Method

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2.1 An illustrative example 2.2 Solution given by some previous analytic techniques 2.2.1 Perturbation method 2.2.2 Lyapunov's artificial small parameter method 2.2.3 Adomian's decomposition method 2.2.4 The δ-expansion method 2.3 Homotopy analysis solution 2.3.1 Zero-order deformation equation 2.3.2 High-order deformation equation 2.3.3 Convergence theorem 2.3.4 Some fundamental rules 2.3.5 Solution expressions 2.3.6 The role of the auxiliary parameter 2.3.7 Homotopy-Pad´e method

3 Systematic description

3.1 Zero-order deformation equation 3.2 High-order deformation equation 3.3 Convergence theorem 3.4 Fundamental rules 3.5 Control of convergence region and rate 3.5.1 The curve and the valid region 3.5.2 Homotopy-Pade technique 3.6 Further generalization

4 Relations to some previous analytic methods

4.1 Relation to Adomian's decomposition method 4.2 Relation to artificial small parameter method 4.3 Relation to δ-expansion method 4.4 Unification of nonperturbation methods

5 Advantages, limitations, and open questions

5.1 Advantages 5.2 Limitations 5.3 Open questions

6 Simple bifurcation of a nonlinear problem

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7 Multiple solutions of a nonlinear problem

7.1 Homotopy analysis solution 7.1.1 Zero-order deformation equation 7.1.2 High-order deformation equation 7.1.3 Convergence theorem 7.2 Result analysis

8 Nonlinear eigenvalue problem

8.1 Homotopy analysis solution 8.1.1 Zero-order deformation equation 8.1.2 High-order deformation equation 8.1.3 Convergence theorem 8.2 Result analysis

9 Thomas-Fermi atom model

9.1 Homotopy analysis solution 9.1.1 Asymptotic property 9.1.2 Zero-order deformation equation 9.1.3 High-order deformation equations 9.1.4 Recursive expressions 9.1.5 Convergence theorem 9.2 Result analysis

10 Volterra's population model

10.1 Homotopy analysis solution 10.1.1 Zero-order deformation equation 10.1.2 High-order deformation equation 10.1.3 Recursive expression 10.1.4 Convergence theorem 10.2 Result analysis 10.2.1 Choosing a plain initial approximation 10.2.2 Choosing the best initial approximation

11 Free oscillation systems with odd nonlinearity

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12 Free oscillation systems with quadratic nonlinearity

12.1 Homotopy analysis solution 12.1.1 Zero-order deformation equation 12.1.2 High-order deformation equation 12.2 Illustrative examples 12.2.1 Example 12.2.1 12.2.2 Example 12.2.2

13 Limit cycle in a multidimensional system

13.1 Homotopy analysis solution 13.1.1 Zero-order deformation equation 13.1.2 High-order deformation equation 13.1.3 Convergence theorem 13.2 Result analysis

14 Blasius' viscous flow

14.1 Solution expressed by power functions 14.1.1 Zero-order deformation equation 14.1.2 High-order deformation equation 14.1.3 Convergence theorem 14.1.4 Result analysis 14.2 Solution expressed by exponentials and polynomials 14.2.1 Asymptotic property 14.2.2 Zero-order deformation equation 14.2.3 High-order deformation equation 14.2.4 Recursive expressions 14.2.5 Convergence theorem 14.2.6 Result analysis

15 Boundary-layer flows with exponential property

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16 Boundary-layer flows with algebraic property

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17 Von Kŕarmŕan swirling viscous flow

17.1 Homotopy analysis solution 17.1.1 Zero-order deformation equation 17.1.2 High-order deformation equation 17.1.3 Convergence theorem 17.2 Result analysis

18 Nonlinear progressive waves in deep water

18.1 Homotopy analysis solution 18.1.1 Zero-order deformation equation 18.1.2 High-order deformation equation 18.2 Result analysis

Problems and Examples

1. Consider the Duffing equation $\ddot{u} + u + \varepsilon u^3 = 0$. Apply the method of averaging and assume the solution in the form $u(t) = \alpha(t)\cos(t + \beta(t))$. Compare the asymptotic solution with the numerical solution shown in Fig.1

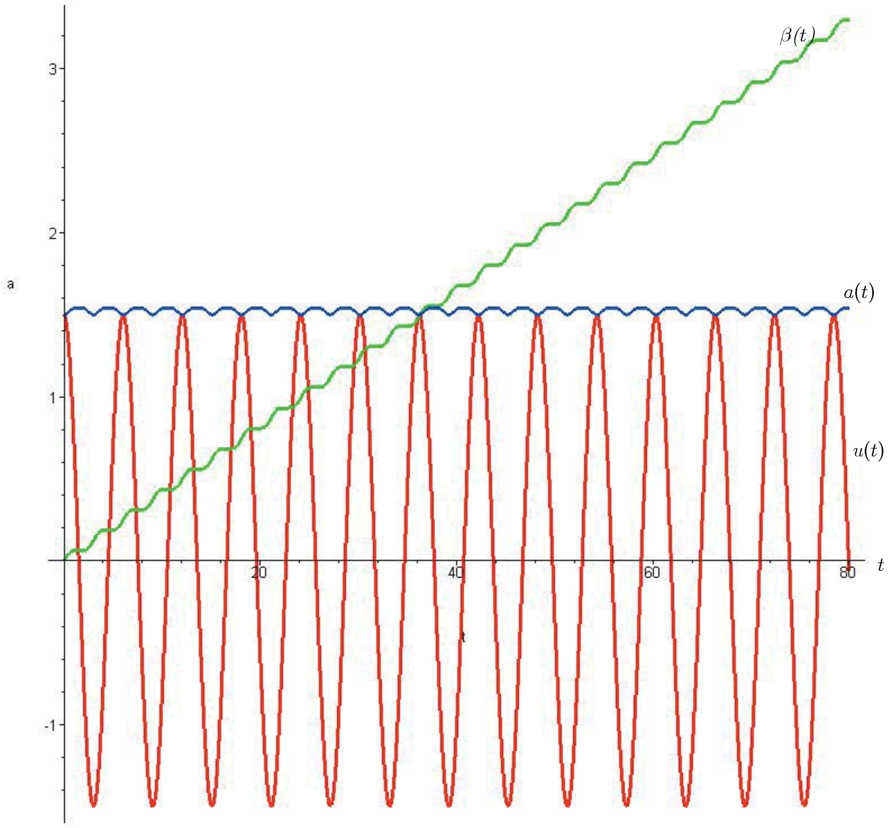

Fig. 1 The Duffing Equation. Numerical Solution

PROBLEMS

For the regular perturbation polynomial problems **1** through **10**: Determine the second order solutions (up to ϵ^2) for the roots.

1 $x^2 + \epsilon x - 1 = 0$

2 $x^2 + (1+\epsilon)x - (2-\epsilon) = 0$

3 $(1+\epsilon)x^2 - 4x + 3 = 0$

4 $(1+\epsilon)x^2 - 3x + (2+\epsilon) = 0$

5 $x^3 - 3x^2 + (2+\epsilon)x + \epsilon = 0$

6 $x^3 + \epsilon x^2 - (4+\epsilon)x + \epsilon = 0$

7 $x^3 + \epsilon x^2 - 7x + (6+\epsilon) = 0$

8 $x^3 + (1+\epsilon)x^2 - (4+\epsilon)x - 4 = 0$

9 $x^4 - 5x^2 + \epsilon x + 4 = 0$

10 $x^4 - 6x^3 + 11x^2 - 6x + \epsilon = 0$

For the singular perturbation problems **11** through **26**: Determine the first three terms of the perturbation series for each of the roots. Note that many of these singular perturbation problems can be transformed into regular perturbation problems via a scale transformation.

11 $x^3 + (1+2\epsilon)x^2 - \epsilon = 0$

12 $x^3 - 2x^2 + (1+\epsilon)x - 2\epsilon = 0$

13 $x^3 + (2+\epsilon)x^2 + (1+\epsilon)x + e = 0$

14 $x^3 - 2x^2 + \epsilon x + 2\epsilon = 0$

15 $x^4 - 2x^3 + (1+\epsilon)x^2 + 2\epsilon x - 4\epsilon = 0$

16 $x^4 + (2+\epsilon)x^3 + x^2 + 2\epsilon x - \epsilon = 0$

17 $x^3 - 3x^2 + (3-\epsilon)x - 1 = 0$

18 $x^3 - (3+2\epsilon)x^2 + 3x + (2\epsilon - 1) = 0$

19 $\epsilon x^3 - x^2 + 1 = 0$

20 $\epsilon x^3 - x + 1 = 0$

21 $\epsilon x^3 - x + 2 + \epsilon = 0$

22 $\epsilon^2 x^3 - \epsilon x^2 - 2x + 2 = 0$

23 $\epsilon^2 x^3 - 3\epsilon x^2 + 2x - 2 = 0$

24 $\epsilon x^4 - x^2 + 3x - 2 = 0$

25 $\epsilon x^4 - x^2 - x + 2 = 0$

26 $\epsilon x^4 - x^2 + 2x - 1 = 0$

1 Show that the general second order linear homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

can be made into a Schrödinger equation with the proper substitution. That is, the y' term can be eliminated.

Hint: Substitute $y = u(x)f(x)$, and determine $u(x)$ so that the equation has no $f'(x)$ term.

2 Show that if $|q(x)| \le M$ for $0 \le x \le x_0$, then by induction,

$$|y_n(x)| \le \frac{M^n x^{2n+1}}{(2n+1)!}$$

for $0 \le x \le x_0$, where $y_0 = x$ and y_n is defined by equation 7.5. From this, show that the perturbation series from example 7.6 converges at x_0 for all ϵ.

3 The Airy equation is given by $y'' - xy = 0$. Use example 7.6, with $q(x) = -x$, to find the first four terms of the perturbation series, using the same initial conditions. How does this compare to the Taylor series solution, done in example 4.18?

4 The parabolic cylinder equation, introduced in example 5.2, is given by $y'' + (v + 1/2 - x^2/4)y = 0$. Use example 7.6, with $q(x) = (2v+1)/2 - x^2/4$, to find the first three terms of the perturbation series, using the same initial conditions.

For problems **5** through **14**: Determine the second order perturbation solutions (up to y_2) for the following initial value problems.

5	$y' + \epsilon e^x y = 0,$	$y(0) = 1$	**10**	$y'' - \epsilon e^x y = 0,$	$y(0) = 1,\ y'(0) = 0$
6	$y' + \epsilon \sin(x) y = 0,$	$y(0) = 2$	**11**	$y'' - \epsilon e^x y = 0,$	$y(0) = 0,\ y'(0) = 1$
7	$y' + \epsilon x y = e^x,$	$y(0) = 1$	**12**	$y'' + \epsilon y' + y = 0,$	$y(0) = 1,\ y'(0) = 0$
8	$y' - \epsilon e^x y = 1,$	$y(0) = 0$	**13**	$y'' + \epsilon x y = x,$	$y(0) = y'(0) = 0$
9	$y' - \epsilon x y = 1/x,$	$y(1) = 0$	**14**	$y'' - \epsilon x y = e^x,$	$y(0) = y'(0) = 1$

For problems **15** through **20**: Determine the second order perturbation solutions (up to y_2) for the following boundary value problems.

15	$y'' - \epsilon x y = 0,$	$y(0) = 0,\ y(1) = 1$
16	$y'' - \epsilon x y = 0,$	$y'(0) = 1,\ y(1) = 1$
17	$y'' + \epsilon x^2 y = 0,$	$y(0) = 0,\ y'(1) = 1$
18	$y'' + \epsilon y' + y = 0,$	$y(0) = 0,\ y(\pi/2) = 1$
19	$y'' + \epsilon x y = x^2,$	$y(0) = y(1) = 0$
20	$y'' + \epsilon y' + y = 2\sin(x),$	$y(0) = y(\pi/2) = 0$

For problems **7** through **12**: Use Van Dyke's method to match the inner and outer solutions up to order ϵ^2 for the following problems. Then find the composite approximations $y_{\text{comp},0}$, $y_{\text{comp},1}$, and $y_{\text{comp},2}$ to the solution to the equation. Note that the inner and outer solutions were found in problems 3 through 14 of section 7.3.

7	$\epsilon y'' + y' + y = 0,$	$y(0) = 1,\ \ y(1) = 2$
8	$\epsilon y'' + 2y' - 2y = 0,$	$y(0) = 2,\ \ y(1) = 1$
9	$\epsilon y'' + (x+1)y' - 2y = 0,$	$y(0) = 0,\ \ y(1) = 4$
10	$\epsilon y'' + (x+1)y' + y = 0,$	$y(0) = 2,\ \ y(1) = 1/2$
11	$\epsilon y'' + y' + 2xy = 0,$	$y(0) = 1,\ \ y(1) = 1$
12	$\epsilon y'' + y' - 2xy = 0,$	$y(0) = 2,\ \ y(1) = 1$