Inverse Problems

Course contents:

The general purpose of this course is to introduce certain key ideas on inverse problems and discuss some meaningful applications. With this approach, the student will be able to study inverse problems and to use them in practical situations. In this course some basic ideas on mathematical modeling is presented. Important *special* types of questions of mathematical modelling and the role of inverse problems in this setting are discussed. The methodology used to illustrate the explanation is multivariate linear regression, and an application to flow in porous medium is considered. The classical analytical difficulties of inverse problems, the questions of existence, uniqueness and ill-posedeness are presented. The discussion is elementary, and the examples come from algebraic equations. The condition on the evaluation of functions is touched upon and the question of stability of numerical algorithms is also pointed out by simple examples. General classifications of mathematical models and inverse problems are presented. The regularization of a finite dimensional inverse problem, in order to get a meaningful approximation of its solution is introduced. Linear algebra ideas, in particular the spectral theory of symmetric matrices, are used to understand the workings of simple regularization schemes. The methods of Tikhonov, steepest descent, Landweber, and conjugate gradient are discussed in this context, and the discrepancy principle is presented.

Subjects covered by the lectures are

- Fundamental Concepts in Inverse Problems
- Spectral Analysis of an Inverse Problem
- Image Restoration
- Radiative Transfer and Heat Conduction
- Thermal Characterization
- Heat Conduction
- Inverse Problems in Fracture Mechanics
- Crack Detection by Scattering of Waves
- Tomographic Evaluation of Materials
- The Reciprocity Gap Functional for Crack detection
- Methods of Solutions to Inverse Problems

Learning outcomes of the course :

Through a deep understanding of the theory and the realization of a project, the student will be able to apply theoretical, asymptotical and numerical tools to inverse problems. In particular:

- He will have a deep understanding of essence of inverse problems and will be able to summary, compare and explain them.
- He will have a deep understanding of the resolution methods of inverse problems, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to apply the resolution methods to classical inverse problems.
- He will be able to analyze and to evaluate (justify and criticise) these methods.
- He will be able to analyze new problems.

Prerequisites and co-requisites/ Recommended optional programme components :

Basic knowledge in

- Differential Equations
- Partial Differential Equations

- Elasticity Theory
- Plasticity Theory

Planned learning activities and teaching methods :

Exercises with professor assistance and personal project.

Mode of delivery (face-to-face ; distance-learning) :

Face-to-Face

Required readings :

- Neto F.D.M., Neto A.J.S. An Introduction to Inverse Problems with Applications. Berlin: Springer, 2013. 255 p.
- Bal G. Introduction to Inverse Problems. New York: Columbia University, 2012. 205 p.
- Muller J.L., Sittanen S. Linear and Nonlinear Inverse Problems with Practical Applications. Helsinki: Computational Science and Engineering, 2012. 372 p.
- Kirsch A. An Introduction to the Mathematical Theory of Inverse Problems, Springer-Verlag, New York, 1996.
- Ammari H. An Introduction to Mathematics of Emerging Biomedical Imaging, vol. 62 of Mathematics and Applications, Springer, New York, 2008.
- Bui H.D. Fracture Mechanics: Inverse problems and Solutions. Dordrecht: Springer, 2006. 376 p.

Assessment methods and criteria :

Evaluation is based on the realization of a project related to the use / development of numerical methods specific to inverse problems and on an examination.
The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked.
Participation to the examination and achievement of the project are mandatory.

Socratic Teaching Method: Class participation is mandatory. Everyone is expected to participate in discussions relating to reading materials, homework, exams and lectures.

Guaranteed Recipe for Success:
1) Take notes during lecture and sections.
2) After each lecture but before the next lecture review your notes. Identify the parts you do not understand.
3) Come to each lecture and discussion section with specific questions.
4) Keep up with the reading so that you have some familiarity with each topic prior to hearing about it in the lecture.
5) Find at least one "partner" in the class with whom you can meet at least once or twice a week to discuss materials from the lectures, the reading assignments and the homework.
6) Take the homework assignment seriously. Do not try to do the whole assignment the night before it is due. Some version of the homework questions will appear on the exams.

INVERSE PROBLEMS
Course Contents

Fundamental notions

Inverse Problems in a Snapshot. AppliedInverseProblems Crossing the Science-Technology Gap.

1 Mathematical Modeling

1.1 Models. 1.2 Observing Reality. 1.3 The Art of Idealization:Model Characterization. 1.4 Resolution of the Idealization: Mathematics. 1.5 Idealized Data Processing: Solution Operator. 1.6 Mind the Gap:Least Squares. 1.7 Optimal Solution. 1.8 A Suboptimal Solution. 1.9 Application to Darcy's Law.

2 Fundamental Concepts in Inverse Problems

2.1 Condition of Function Evaluation. 2.2 Condition as a Derivative Bound. 2.3 Other Derivatives and Other Notions of Condition. 2.4 Condition of Vector-Valued Functions of Several Variables. 2.5 Stability of Algorithms of Function Evaluation. 2.6 Questions on Existence and Uniqueness. 2.6.1 Exact Data. 2.6.2 Real Data. 2.7 Well-Posed Problems. 2.8 Classification of Inverse Problems. 2.8.1 Classes of Mathematical Models. 2.8.2 Classes of Inverse Problems.

3 Spectral Analysis of an Inverse Problem

3.1 An Example. 3.2 Uncertainty Multiplication Factor. 3.3 Regularization Scheme. 3.4 Tikhonov's Regularization. 3.5 Regularization Strategy. 3.6 Reference Value in Tikhonov's Method. 3.7 Steepest Descent Method. 3.8 Landweber Method. 3.9 Discrepancy Principle. 3.10 Conjugate Gradient. 3.10.1 Conjugate Gradient Algorithm. 3.10.2 Two-Step Optimization. 3.10.3 A Few Geometric Properties. 3.11 Spectral Analysis of Tikhonov's Regularization.

4 Image Restoration

4.1 Degraded Images. 4.2 Restoring Images. 4.3 Restoration Algorithm. 4.3.1 Newton's Method. 4.3.2 Modified Newton's Method with Gain Factor. 4.3.3 Stopping Criterion. 4.3.4 Regularization Strategy. 4.3.5 Gauss-Seidel Method. 4.3.6 Restoration Methodology. 4.4 PhotoRestoration. 4.5 Text Restoration. 4.6 Biological Image Restoration.

5 Radiative Transfer and Heat Conduction

5.1 Mathematical Description. 5.2 Modified Newton's Method. 5.3 Levenberg-Marquardt's Method. 5.4 Confidence Intervals for Parameter Estimates. 5.5 Phase Function, Albedo and Optical Thickness. 5.6 Thermal Conductivity, Optical Thickness and Albedo. 5.7 Refractive Index and Optical Thickness.

6. Thermal Characterization

6.1 Experimental Device: Hot Wire Method. 6.2 Traditional Experimental Approach. 6.3 Inverse Problem Approach. 6.3.1 Heat Equation. 6.3.2 Parameter Estimation. 6.3.3 Levenberg-Marquardt. 6.3.4 Confidence Intervals. 6.3.5 Application to the Characterization of a Phenolic Foam with Lignin. 6.4 Experiment Design.

7 Heat Conduction

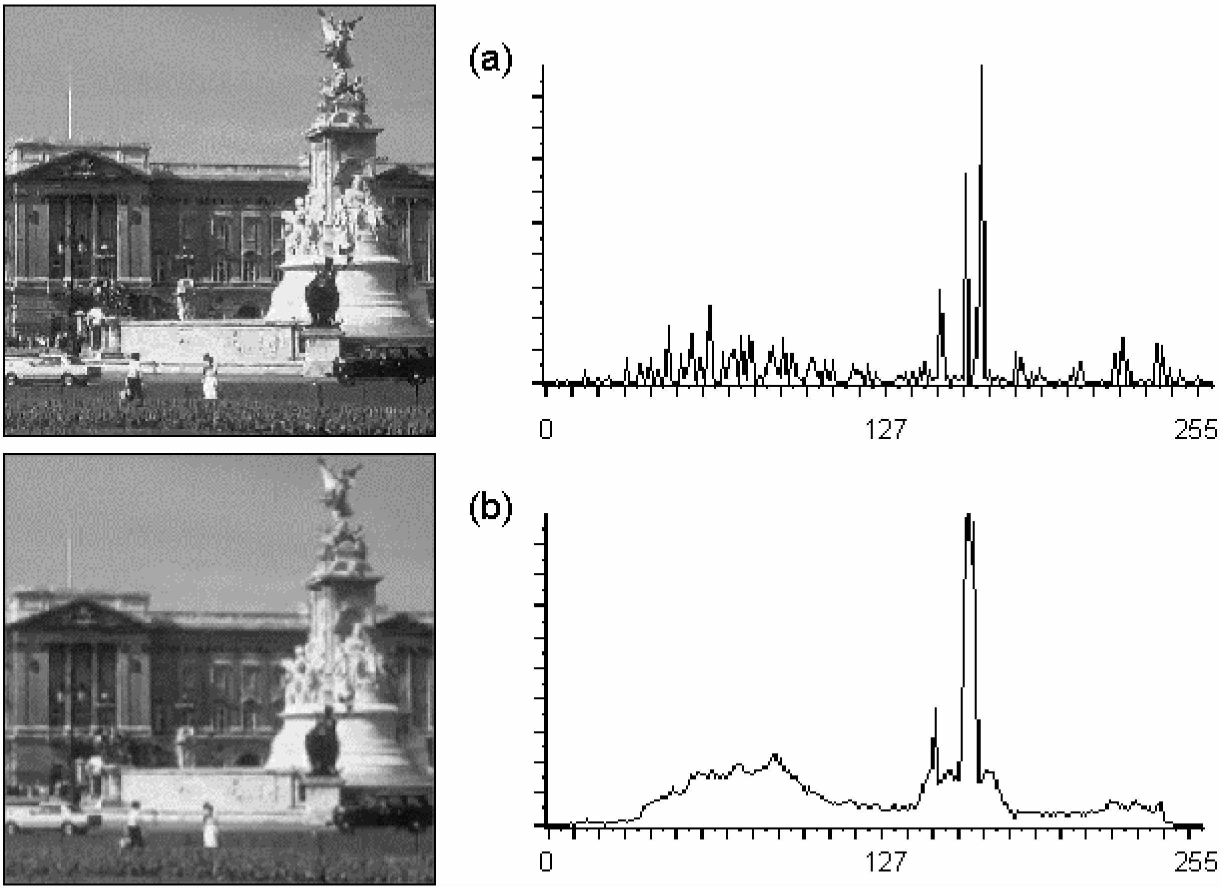
7.1 Mathematical Formulation. 7.2 Expansion in Terms of Known Functions. 7.3 Conjugate Gradient Method. 7.4 Thermal Source in Heat Transfer by Conduction. 7.4.1 Heat Transfer by Conduction. 7.4.2 Sensitivity Problem. 7.4.3 Adjoint Problem and Gradient Equation. 7.4.4 Computation of the Critical Point. 7.5 Minimization with the Conjugate Gradient Method. 7.5.1 Conjugate Gradient Algorithm. 7.5.2 Stopping Criterion and Discrepancy Principle. 7.6 Estimation Results.

8 A General Perspective

8.1 Inverse Problems and Their Types. 8.1.1 Classification of Problems. 8.1.2 Observation Operators. 8.1.3 Inverse Problems in Practice. 8.1.4 Domain of Dependence. 8.1.5 Types of Inverse Problems. 8.2 Gradient Calculation. 8.2.1 Reformulation of the Direct Problem. 8.2.2 Gradient Calculation by Its Definition. 8.2.3 Interpretation of the Gradient. 8.2.4 Operational Rule for the Computation of the Gradient.

Some Notions on Model Complexity and Knowledge

Spectral Theory and a Few Other Ideas From Mathematics



Restoration of an artificially modified image. Images are on the left and their shade of gray histograms are on the right. (a) original image; (b) blurred image. (Author: G. A. G. Cidade from the Universidade Federal do Rio de Janeiro).

Lecture. Methods of Solutions to Inverse Problems

Methods of Solution to Inverse Problems

Inverse problems consist, in principle, in inverting the relation of *cause* (**z**) and *effect* which is considered as data (**d**), *accessible* to experiments. It is necessary that data are accessible to experiments, otherwise there is no significant inverse problem. Such a definition should give a status of inverse problems to any forward problems : for instance find the internal stress in an elastic solid (the cause) from measured boundary displacement (data), or find the internal displacement from measured stress vectors on the boundary. Both forward problems are well-posed ones, in the sense that the relation of cause and effect is well established by the equilibrium equations of elasticity which has good mathematical property for usual elastic body. There is only one condition for the second problem to have a solution, that is the stress vector on the boundary is a self equilibrated one in the absence of body force. Well-posed problem has a unique solution which depends continuously on the data. The main characteristics of inverse problems are twofold: first, the relation of cause to effect in real experiments is often not completely known, second, even when this relation is well established by some mapping A

$$\mathbf{z} \to A(\mathbf{z}) = \mathbf{d} \qquad (1.1)$$

of metric space Z into D, the problem (1.1) is *ill-posed* (not well-posed).

1.1 The ill-posedness of the inverse problem

Let us reconsider the example of the earthquake inverse problem of the preceding chapter to get an inside into inverse problems. We assume that a model exists for the mapping A between the unknown **z**={ Σ (t), τ (**x**).(**x**,t).**n**}, and the acceleration data d^{meas}(**x**,t) on [0,T]×$\partial\Omega$ ($\partial\Omega$ consists of the ground G and the internal half-sphere H).
A mapping A can be defined as the elastic response of the solid Ω to the shear stress and the stress free condition on $\partial\Omega$. Solution to
the earthquake inverse problem generally consists in the minimization

$$\underset{\mathbf{z}}{\mathrm{Min}} \int_{\partial\Omega} \int_0^T \left\| d(\Sigma(t), \tau, \ \tau.\sigma.\mathbf{n}; t) - d^{meas}(t) \right\|^2 dtdS \qquad (1.2)$$

where we have listed all arguments **z** of the predicted data and the measured one d^{meas}(t) on the ground. If data are only measured on the ground, missing data on the remaining part of the boundary make it impossible to solve the problem (1.2). Thus only an approximate solution can be expected. When data are known on the whole boundary G∪H, the solution exists and is unique only for *exact* data d^{exact}(t), in the sense stated in the preceding chapter, that is

$$d^{exact}(t) \equiv A(\mathbf{z}).$$

for some **z**. Mathematically, data d^{exact}(t) must belong to the domain D(A) of operator A. Finally in the ideal case of an exact data d^{exact}(t), the least square method of fitting between approximate prediction and measured data uniformly distributes errors of the modelling on the whole space-time. Needless to say, with approximate solution and noised data, one cannot expect a good solution. "*Even if the fitting of data seems to be quite good, the faulting process is poorly reproduced, so that in the real case, it would be difficult to know when one has obtained the correct solution*", as noted by Das and Suhadolc (1996). This is why a best fitting of the prediction to data does not guarantee that the solution is a correct one. This happens when the functional (12.2) has a flat minimum.
Consider again the earthquake problem and examine the kinematic approach actually considered in Geophysics which consists in modelling the displacement response on the ground **u**(**x**,t) from the unknown slip γ (**y**,t) on the unknown fault Σ (t), (Das and Kostrov, 1990)

$$u_k(\mathbf{x}, t) = \int_0^t d\tau \int_{\Sigma(\tau)} K_{ki}(\mathbf{x}, \mathbf{y}, t, \tau)\gamma_i(\mathbf{y}, \tau) dS_y \qquad (1.3)$$

where K is the impulse response of the stress free solid $\partial\Omega$ (approximated by an infinite elastic medium). Even for known Σ (t), Eq. (1.3) to determine χ from measurement of **u** (after a double time and spatial integration of the data) is well known to be unstable. Moreover, no mathematical results exist for the equation with the unknowns **u** and Σ (t). Another form of Eq. (1.3) with the integration on the fault plane P is

$$u_k(\mathbf{x}, t) = \int_0^t d\tau \int_P K_{ki}(\mathbf{x}, \mathbf{y}, t, \tau)\gamma_i(\mathbf{y}, \tau)\phi(\mathbf{y}, \tau) dS_y \qquad (1.4)$$

with the characteristic function $\phi(\mathbf{y}, \tau)$ in the fault plane P. Even for known P, the simpler form equation (1.4) is *non linear*. Therefore, in order to effectively solve (1.4) one makes use of a priori assumptions on the geometry of Σ (t), its velocity $\dot{\Sigma}(\tau)$, the slip direction etc. An a priori knowledge reduces the number of the unknowns. It is thus important to get such an information from the physics of the problem considered. Mathematically, one can also reduce the number of unknowns by considering subspaces, or by using non physical prior bounds such as minimum norm solution, smooth solution etc. The best method, however, is to get exact mathematical and physical constraints between solutions and data, as that provided by the RG functional method , rather than artificial and non physical constraints.

1.2 General considerations on inverse problems

Consider two metric spaces, the space Z of unknown model parameters **z** and the space of observable quantitities or measured data **d**. The so-called observation equation is

$$A(\mathbf{z}) = \mathbf{d} \quad (1.5)$$

Fig. 1.1 shows a schematic view of spaces and mappings, $A(Z)\subset D$ is the range of A, **d** is a given data which generally does not belongs to A(Z), $\mathbf{d}^1 \in A(Z)\subset D$ is the nearest point $\mathbf{d}^1$ of minimum distance

$$\mathbf{d}^1 = \arg\min_{d'} \rho(\mathbf{d}, \mathbf{d}') \quad (1.6)$$

$$\mathbf{d}^1 = \mathbf{P}_{A(Z)}(\mathbf{d}) \quad (1.7)$$

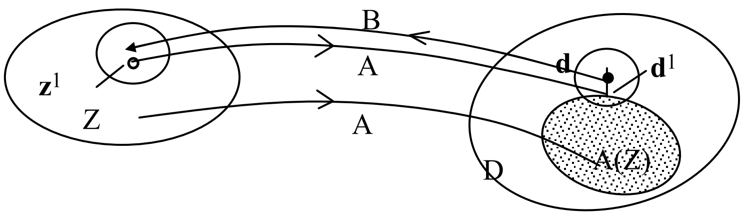


Figure 1.1: The model space Z and data space D: The quasi-solution $\mathbf{z}^1$ is such that $A(\mathbf{z}^1)$ is the projection $\mathbf{d}^1=\mathbf{P}_{A(Z)}(\mathbf{d})$ of the data **d** on $A(Z)\subset D$

where $\mathbf{P}_{A(Z)}$ is the projector in A(Z). Any $\mathbf{z}^1$ such that $A(\mathbf{z}^1)=d^1$ is called a *quasi-solution* of Eq. (1.5). To construct such as solution, Tikhonov and Arsenine (1977) proposed to seek a mapping B of D into Z, such that B(**d**) is a quasi solution $A(B(\mathbf{d}))=\mathbf{d}^1$ and that B(**d**) is sufficiently regular and stable with respect to the perturbation of data. There is no guarantee for the uniqueness of the mapping B. In practice, the conditions for the solvability of (1.5) are

a) One has a priori knowledge of the subspace Z_1 of solutions for some class of regular data in A(Z). For instance one knows that the solution is near the point $\mathbf{z}^0$ (a priori information) belonging to Z_1,

b) The solution is unique in Z_1,

c) The solution is continuous in $\mathbf{d}\in A(Z_1)$.

Tikhonov and Arsenine (1977) proved another stronger result. For compact Z_1, under assumptions a) and b), there exists a continuous, monotonous and non decreasing function $\alpha(\varepsilon)$, $\alpha(0)=0$ such that

$$\rho(A\mathbf{z}^1, A\mathbf{z}^2)_D \le \varepsilon \Rightarrow \rho(\mathbf{z}^1, \mathbf{z}^2)_A \le \alpha(\varepsilon) \quad (1.8)$$

This inference means that condition c) is fulfilled as long as a) and b) are.

It is essential that some knowledge of subspace Z_1 exists. For instance, in the numerical inversion of seismic data, approximate solution is often stabilized by considering prior bounds on the fault velocity, such as constant velocity, knowledge on the planar nature of the fault – an assumption frequently used – physical constraints on the constant slip direction τ, neither back slip in the time interval considered nor variation of the slip direction etc. Artificial data, generated for an ideal faulting model with known $\Sigma(t)$ and shear release history, are used for testing the inversion procedure, (Das and Suhadolc, 1996). It is found that, if subspace Z_1 is too small, there appears a ghost solution together with the approximate one. This phenomenon should be linked to the existence of local minima of the flat least square functional (1.2).