# Nonlinear Dynamics, Chaos and Fractals

**Course contents:**

This course is aimed at newcomers to nonlinear dynamics and chaos, especially students taking a first course in the subject. The presentation stresses analytical methods, concrete examples and geometric intuition. The theory is developed systematically, starting with first-order differential equations and their bifurcations, followed by phase plane analysis, limit cycles and their bifurcations, and culminating with the Lorenz equations, chaos, iterated maps, period doubling, renormalization, fractals, and strange attractors. A unique feature of the course is its emphasis on applications. These include mechanical vibrations, lasers, biological rhythms, superconducting circuits, insect outbreaks, chemical oscillators, genetic control systems, chaotic waterwheels, and even a technique for using chaos to send secret messages. In each case, the scientific background is explained at an elementary level and closely integrated with the mathematical theory.

**Subjects covered by the lectures are**

- Brief Literature Overview.
- First-Order Differential Equations and Their Bifurcations
- Phase Plane Analysis, Limit cycles and Their Bifurcations
- The Lorenz Equations Chaos, Iterated Maps, Period Doubling, Renormalization, Fractals, and Strange Attractors.
- Applications: Mechanical Vibrations, Lasers, Biological Rhythms, Superconducting Circuits, Insect outbreaks, Chemical Oscillators, Genetic Control Systems, Chaotic waterwheels

**Learning outcomes of the course :**

Through a deep understanding of the theory and the realization of a project, the student will be able to know the fundamentals of the theory of dynamical systems. In particular:

- He will have a deep understanding of the theory of dynamical systems and will be able to explain, summary, compare and elucidate the basics of the theory.
- He will have a deep understanding of the resolution methods of the problems of the theory of dynamical systems, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to know a basic understanding of notions of Nonlinear Dynamics and Deterministic Chaos and the special cases of chaotic dynamics in Hamiltonian Systems and Dissipative Dynamical Systems.
- He will be able to analyze and to evaluate (justify and criticise) the methods of Nonlinear Dynamics.
- He will be able to analyze new problems.

**Prerequisites and co-requisites/ Recommended optional programme components :**

Basic knowledge in

- Differential Equations
- Elasticity Theory

**Planned learning activities and teaching methods :**

Exercises with professor assistance and personal project.

**Mode of delivery (face-to-face ; distance-learning) :**

Face-to-Face

**Required readings :**

- Acary, V., Brogliato, B.: Numerical methods for nonsmooth dynamical systems: Applications in Mechanics and Electronics. Heidelberg: Springer, 2008.
- Pilipchuk V.N. Nonlinear Dynamics. Between Linear and Impact Limits. Heidelberg, Berlin: Springer-Verlag, 2010. 336 p.
- Kryloff, N., Bogoliuboff, N.: Introduction to Non-Linear Mechanics. Princeton: Princeton University Press, 1943.
- Peitgen H.O., Jurgen H., Saupe D. Chaos and Fractals. New Frontiers of Science. Berlin: Springer, 2004.
- Korsch H.J., Jodl H.-J., Hartmann T. Chaos. New York: Springer, 2008. 353 p.

**Assessment methods and criteria :**

Evaluation is based on the realization of a project related to the use / development of numerical methods specific to nonlinear dynamics and on an examination.
The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked.
Participation to the examination and achievement of the project are mandatory.

**Socratic Teaching Method:** Class participation is mandatory. Everyone is expected to participate in discussions relating to reading materials, homework, exams and lectures.

**Guaranteed Recipe for Success:**
1) Take notes during lecture and sections.
2) After each lecture but before the next lecture review your notes. Identify the parts you do not understand.
3) Come to each lecture and discussion section with specific questions.
4) Keep up with the reading so that you have some familiarity with each topic prior to hearing about it in the lecture.
5) Find at least one "partner" in the class with whom you can meet at least once or twice a week to discuss materials from the lectures, the reading assignments and the homework.
6) Take the homework assignment seriously. Do not try to do the whole assignment the night before it is due. Some version of the homework questions will appear on the exams.

**Nonlinear Dynamics, Chaos and Fractals**
**Course Contents**

1.1 Brief Literature Overview. 1.2 Asymptotic Meaning of the Approach. 1.2.1 Two Simple Limits of Lyapunov Oscillator. 1.2.2 Oscillating Time and Hyperbolic Numbers, Standard and Idempotent Basis. 1.3 Quick ‘Tutorial’. 1.3.1 Remarks on the Basic Functions. 1.3.2 Viscous Dynamics under the Sawtooth Forcing. 1.3.3 The Rectangular Cosine Input. 1.3.4 Oscillatory Pipe Flow Model. 1.3.5 Periodic Impulsive Loading. 1.3.6 Strongly Nonlinear Oscillator. 1.4 Geometrical Views on Nonlinearity. 1.4.1 Geometrical Example. 1.4.2 Nonlinear Equations and Nonlinear Phenomena. 1.4.3 Rigid-Body Motions and Linear Systems. 1.4.4 Remarks on the Multi-dimensional Case. 1.4.5 Elementary Nonlinearities. 1.4.6 Example of Simplification in Nonsmooth Limit. 1.4.7 Non-smooth

Time Arguments. 1.4.8 Further Examples and Discussion. 1.4.9 Differential Equations of Motion and Distributions. 1.5 Non-smooth Coordinate Transformations. 1.5.1 Caratheodory Substitution. 1.5.2 Transformation of Positional Variables. 1.5.3 Transformation of State Variables.

**2 Smooth Oscillating Processes**

2.1 Linear and Weakly Non-linear Approaches. 2.2 A Brief Overview of Smooth Methods. 2.2.1 Periodic Motions of Quasi Linear Systems. 2.2.2 The Idea of Averaging. 2.2.3 Averaging Algorithm for Essentially Nonlinear Systems. 2.2.4 Averaging in Complex Variables. 2.2.5 Lie Group Approaches.

**3 Nonsmooth Processes as Asymptotic Limits**

3.1 Lyapunov' Oscillator. 3.2 Nonlinear Oscillators Solvable in Elementary Functions. 3.2.1 Hardening Case. 3.2.2 Localized Damping. 3.2.3 Softening Case. 3.3 Nonsmoothness Hiden in Smooth Processes. 3.3.1 Nonlinear Beats Model. 3.4 Nonlinear Beat Dynamics: The Standard Averaging Approach. 3.4.1 Asymptotic of Equipartition. 3.4.2 Asymptotic of Dominants. 3.4.3 Necessary Condition of Energy Trapping. 3.4.4 Sufficient Condition of Energy Trapping. 3.5 Transition from Normal to Local Modes. 3.6 System Description. 3.7 Normal and Local Mode Coordinates. 3.8 Local Mode Interaction Dynamics. 3.9 Auto-localized Modes in Nonlinear Coupled Oscillators.

**4 Nonsmooth Temporal Transformations (NSTT)**

4.1 Non-smooth Time Transformations. 4.1.1 Positive Time. 4.1.2 'Single-Tooth' Substitution. 4.1.3 'Broken Time' Substitution. 4.1.4 Sawtooth Sine Transformation. 4.1.5 Links between NSTT and Matrix Algebras. 4.1.6 Differentiation and Integration Rules. 4.1.7 NSTT Averaging. 4.1.8 Generalizations on Asymmetrical Sawtooth Wave. 4.1.9 Multiple Frequency Case. 4.2 Idempotent Basis Generated by the Triangular Sine-Wave. 4.2.1 Definitions and Algebraic Rules. 4.2.2 Time Derivatives in the Idempotent Basis. 4.3 Idempotent Basis Generated by Asymmetric Triangular Wave. 4.3.1 Definition and Algebraic Properties. 4.3.2 Differentiation Rules. 4.3.3 Oscillators in the Idempotent Basis. 4.3.4 Integration in the Idempotent Basis. 4.4 Discussions, Remarks and Justifications. 4.4.1 Remarks on Nonsmooth Solutions in the Classical Dynamics. 4.4.2 Caratheodory Equation. 4.4.3 Other Versions of Periodic Time Substitutions. 4.4.4 General Case of Non-invertible Time and Its Physical Meaning. 4.4.5 NSTT and Cnoidal Waves.

**5 Sawtooth Power Series**

5.1 Manipulations with the Series. 5.1.1 Smoothing Procedures. 5.2 Sawtooth Series for Normal Modes. 5.2.1 Periodic Version of Lie Series. 5.3 Lie Series of Transformed Systems. 5.3.1 Second-Order Non-autonomous Systems. 5.3.2 NSTT of Lagrangian and Hamiltonian Equations. 5.3.3 Remark on Multiple Argument Cases.

**6 NSTT for Linear and Piecewise-Linear Systems**

6.1 Free Harmonic Oscillator: Temporal Quantization of Solutions. 6.2 Non-autonomous Case. 6.2.1 Standard Basis. 6.2.2 Idempotent Basis. 6.3 Systems under Periodic Pulsed Excitation. 6.3.1 Regular Periodic Impulses. 6.3.2 Harmonic Oscillator under the Periodic Impulsive Loading. 6.3.3 Periodic Impulses with a Temporal 'Dipole' Shift. 6.4 Parametric Excitation. 6.4.1 Piecewise-Constant Excitation. 6.4.2 Parametric Impulsive Excitation. 6.4.3 General Case of Periodic Parametric Excitation. 6.5 Input-Output Systems. 6.6 Piecewise-Linear Oscillators with Asymmetric Characteristics. 6.6.1 Amplitude-Phase Equations. 6.6.2 Amplitude Solution. 6.6.3 Phase Solution. 6.6.4 Remarks on Generalized Taylor Expansions. 6.7 Multiple Degrees-of-Freedom Case. 6.8 The Amplitude-Phase Problem in the Idempotent Basis.

**7 Periodic and Transient Nonlinear Dynamics under Discontinuous Loading**

7.1 Nonsmooth Two Variables Method. 7.2 Resonances in the Duffing's Oscillator under Impulsive Loading. 7.3 Strongly Nonlinear Oscillator under Periodic Pulses. 7.4 Impact Oscillators under Impulsive Loading.

**8 Strongly Nonlinear Vibrations**

8.1 Periodic Solutions for First Order Dynamical Systems. 8.2 Second Order Dynamical Systems. 8.3 Periodic Solutions of Conservative Systems. 8.3.1 The Vibroimpact Approximation. 8.3.2 One Degree-of-Freedom General Conservative Oscillator. 8.3.3 A Nonlinear Mass-Spring Model That Becomes Linear at High Amplitudes. 8.3.4 Strongly Non-linear Characteristic with a Step-Wise

Discontinuity at Zero. 8.3.5 A Generalized Case of Odd Characteristics. 8.4 Periodic Motions Close to Separatrix Loop.
8.5 Self-excited Oscillator. 8.6 Strongly Nonlinear Oscillator with Viscous Damping. 8.6.1 Remark on NSTT Combined with Two Variables Expansion. 8.6.2 Oscillator with Two Nonsmooth Limits. 8.7 Bouncing Ball. 8.8 The Kicked Rotor Model. 8.9 Oscillators with Piece-Wise Nonlinear Restoring Force Characteristics.

**9 Strongly Nonlinear Waves**.

9.1 Wave Processes in One-Dimensional Systems. 9.2 Klein-Gordon Equation.

**10 Impact Modes and Parameter Variations**.

10.1 An Introductory Example. 10.2 Parameter Variation and Averaging. 10.3 A Two-Degrees-of-Freedom Model. 10.4 Averaging in the 2DOF System. 10.5 Impact Modes in Multiple Degrees of Freedom Systems. 10.5.1 A Double-Pendulum with Amplitude Limiters. 10.5.2 A Mass-Spring Chain under Constraint Conditions. 10.6 Systems with Multiple Impacting Particles.

**11 Principal Trajectories of Forced Vibrations**

11.1 Introductory Remarks. 11.2 Principal Directions of Linear Forced Systems. 11.3 Definition for Principal Trajectories of Nonlinear Discrete Systems. 11.4 Asymptotic Expansions for Principal Trajectories. 11.5 Definition for Principal Modes of Continuous Systems.

**12 NSTT and Shooting Method for Periodic Motions**

12.1 Introductory Remarks. 12.2 Problem Formulation. 12.3 Sample Problems and Discussion. 12.3.1 Smooth Loading. 12.3.2 Step-Wise Discontinuous Input. 12.3.3 Impulsive Loading. 12.4 Other Applications. 12.4.1 Periodic Solutions of the Period - $n$. 12.4.2 Two-Degrees-of-Freedom Systems. 12.4.3 The Autonomous Case.

**13 Essentially Non-periodic Processes**

13.1 Nonsmooth Time Decomposition and Pulse Propagation in a Chain of Particles. 13.2 Impulsively Loaded Dynamical Systems. 13.2.1 Harmonic Oscillator under Sequential Impulses. 13.2.2 Random Suppression of Chaos.

**14 Spatially-Oscillating Structures**

14.1 Periodic Nonsmooth Structures. 14.2 Averaging for One-Dimensional Periodic Structures. 14.3 Two Variable Expansions. 14.4 Second Order Equations. 14.5 Acoustic Waves from Non-smooth Periodic Boundary Sources. 14.6 Spatio-temporal Periodicity. 14.7 Membrane on a Two-Dimensional Periodic Foundation. 14.8 The Idempotent Basis for Two-Dimensional Structures.

**15 Nonlinear Dynamics and Deterministic Chaos**

15.1 Deterministic Chaos. 15.2 Hamiltonian Systems. 15.2.1 Integrable and Ergodic Systems. 15.2.2 Poincare Sections. 15.2.3 The KAM Theorem. 15.2.4 Homoclinic Points. 15.3 Dissipative Dynamical Systems. 15.3.1 Attractors. 15.3.2 Routes to Chaos. 15.4 Special Topics. 15.4.1 The Poincarre-Birkhoff Theorem. 15.4.2 Continued Fractions. 15.4.3 The Lyapunov Exponent. 15.4.4 Fixed Points of One-Dimensional Maps. 15.4.5 Fixed Points of Two-Dimensional Maps. 15.4.6 Bifurcations.

**16 Billiard Systems**

16.1 Deformations of a Circle Billiard. 16.2 Numerical Techniques. 16.3 Interacting with the Program. 16.4 Computer Experiments. 16.4.1 From Regularity to Chaos. 3.4.2 Zooming In. 3.4.4 Suggestions for Additional Experiments. 3.5 Suggestions for Further Studies. 3.6 Real Experiments and Empirical Evidence.

**17 Gravitational Billiards: The Wedge**

17.1 The Poincarre Mapping. 17.2 Interacting with the Program. 17.3 Computer Experiments. 17.3.1 Periodic Motion and Phase Space Organization. 17.3.2 Bifurcation Phenomena. 17.3.3 'Plane Filling' Wedge Billiards. 17.3.4 Suggestions for Additional Experiments. 17.4 Suggestions for Further Studies. 17.5 Real Experiments and Empirical Evidence.

**18 The Double Pendulum**

18.1 Equations of Motion. 18.2 Numerical Algorithms.18.3 Interacting with the Program. 18.4 Computer Experiments. 18.4.1 Different Types of Motion. 18.4.2 Dynamics of the Double Pendulum. 18.4.3 Destruction of Invariant Curves. 18.4.4 Suggestions for Additional Experiments. 18.5 Real Experiments and Empirical Evidence.

**19 Chaotic Scattering**

19.1 Scattering off Three Disks. 19.2 Numerical Techniques. 19.3 Interacting with the Program. 19.4 Computer Experiments. 19.4.1 Scattering Functions and Two-Disk Collisions. 19.4.2 Tree Organization of Three-Disk Collisions. 19.4.3 Unstable Periodic Orbits. 19.4.4 Fractal Singularity Structure. 19.4.5 Suggestions for Additional Experiments. 19.5 Suggestions for Further Studies. 19.6 Real Experiments and Empirical Evidence.

**20 Fermi Acceleration**

20.1 Fermi Mapping. 20.2 Interacting with the Program. 20.3 Computer Experiments. 20.3.1 Exploring Phase Space for Different Wall Oscillations. 20.3.2 KAM Curves and Stochastic Acceleration. 20.3.3 Fixed Points and Linear Stability. 20.3.4 Absolute Barriers. 20.3.5 Suggestions for Additional Experiments. 20.4 Ideas and Suggestions for Further Studies and Considerations. 20.5 Real Experiments and Empirical Evidence.

**21 The Duffing Oscillator**

21.1 The Duffing Equation. 21.2 Numerical Techniques. 21.3 Interacting with the Program. 21.4 Computer Experiments. 21.4.1 Chaotic and Regular Oscillations. 21.4.2 The Free Duffing Oscillator. 21.4.3 Anharmonic Vibrations: Resonances and Bistability. 21.4.4 Coexisting Limit Cycles and Strange Attractors. 21.4.5 Suggestions for Additional Experiments. 21.5 Suggestions for Further Studies. 21.6 Real Experiments and Empirical Evidence.

**22 Feigenbaum Scenario**

22.1 One-Dimensional Maps. 22.2 Interacting with the Program. 22.3 Computer Experiments. 22.3.1 Period-Doubling Bifurcations. 22.3.2 The Chaotic Regime. 22.3.3 Lyapunov Exponents. 22.3.4 The Tent Map. 22.3.5 Suggestions for Additional Experiments. 22.4 Suggestions for Further Studies. 22.5 Real Experiments and Empirical Evidence.

**23 Nonlinear Electronic Circuits**

23.1 A Chaos Generator. 23.2 Numerical Techniques. 23.3 Interacting with the Program. 23.4 Computer Experiments. 23.4.1 Hopf Bifurcation. 23.4.2 Period-Doubling. 23.4.3 Return Map. 23.4.4 Suggestions for Additional Experiments. 23.5 Real Experiments and Empirical Evidence.

**24 Mandelbrot and Julia Sets**

24.1 Two-Dimensional Iterated Maps. 24.2 Numerical Methods. 24.3 Interacting with the Program. 24.4 Computer Experiments. 24.4.1 Mandelbrot and Julia-sets. 24.4.2 Zooming into the Mandelbrot Set. 24.4.3 General Two-Dimensional Quadratic Mappings. 24.4.4 Suggestions for Additional Experiments. 24.5 Suggestions for Further Studies. 24.6 Real Experiments and Empirical Evidence.

**25Ordinary Differential Equations**

25.1 Numerical Techniques. 25.2 Interacting with the Program.25.3 Computer Experiments. 25.3.1 The Pendulum. 25.3.2 A Simple Hopf Bifurcation. 25.3.3 The Duffing Oscillator Revisited. 25.3.4 Hill's Equation. 25.3.5 The Lorenz Attractor. 25.3.6 The Rossler Attractor. 25.3.7 The Hrenon-Heiles System. 25.3.8 Suggestions for Additional Experiments. 25.4 Suggestions for Further Studies.

**26 Kicked Systems**

26.1 Interacting with the Program. 26.2 Computer Experiments. 26.2.1 The Standard Mapping. 26.2.2 The Kicked Quartic Oscillator. 26.2.3 The Kicked Quartic Oscillator with Damping. 26.2.4 The Hrenon Map. 26.2.5 Suggestions for Additional Experiments. 26.3 Real Experiments and Empirical Evidence.

M. Tabor, *Chaos and Integrability in Nonlinear Dynamics*, John Wiley, New York, 1989

EXAMPLES. Observe The Mandelbrot-set.

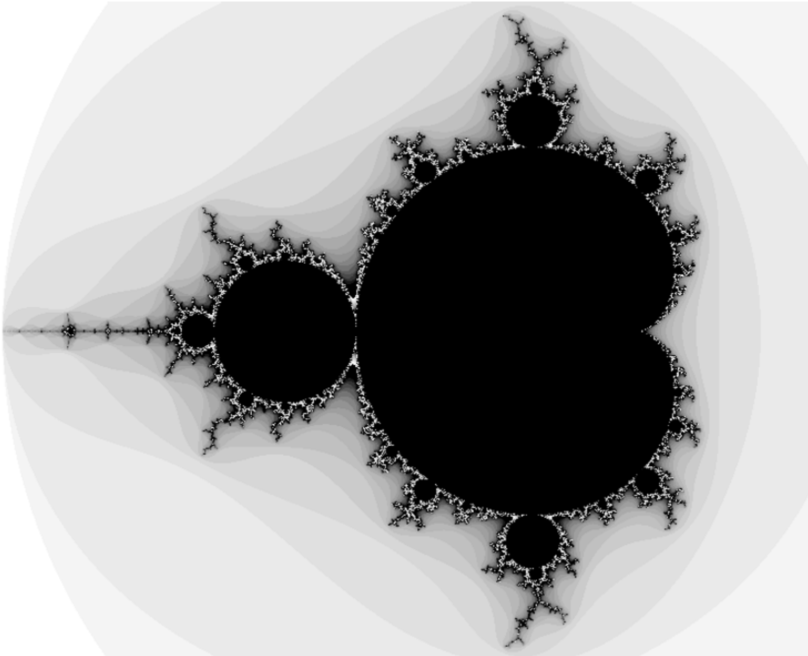
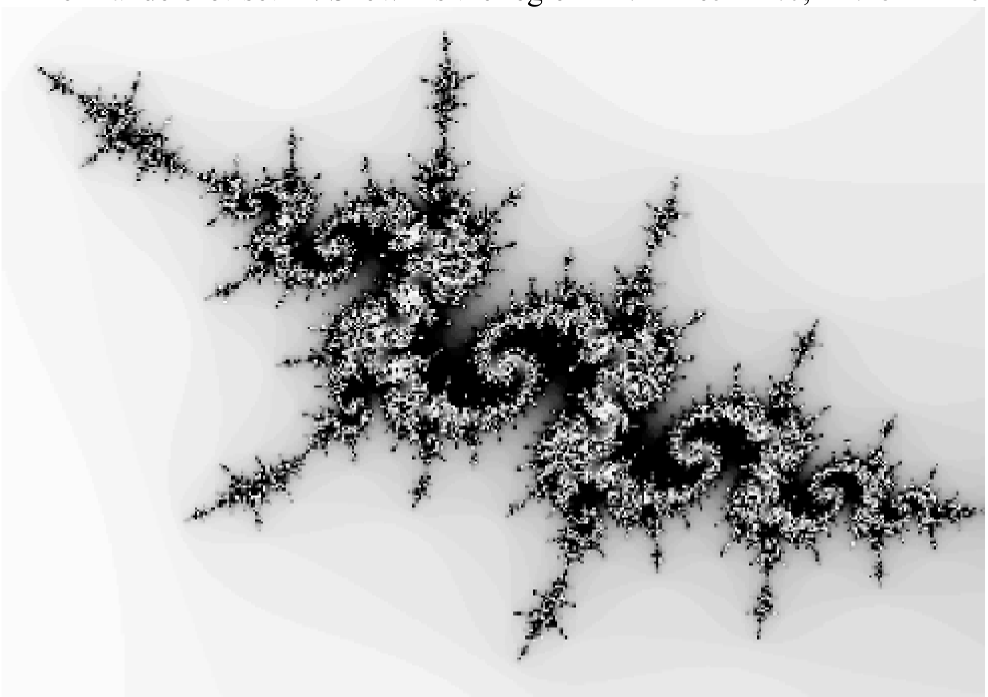


Figure 1 The Mandelbrot-set $M$. Shown is the region $-2.1 < \mathrm{Re}\,c < 1.0$, $-1.15 < \mathrm{Im}\,c < 1.15$



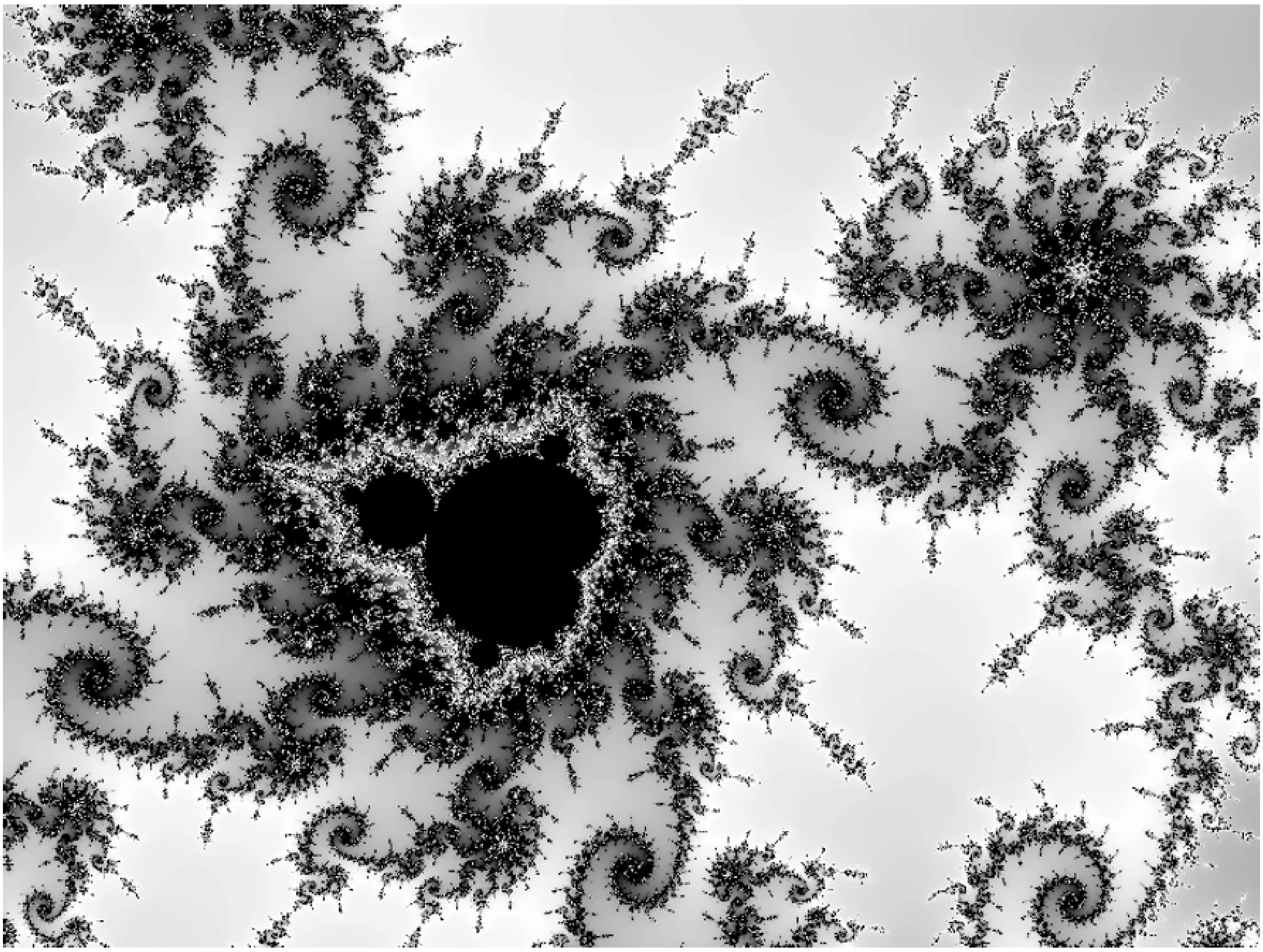


Figure 2. A magnification of the Mandelbrot-set $M$ shows a microscopic replica of the whole set. Shown is a tiny region $-1.2606084 < \mathrm{Re}c < -1.2604275$, $-0.04045689 < \mathrm{Im}c < -0.04032127$ of Figure 1