PLASTICITY THEORY

Course contents:

The class presents the fundamentals of plasticity theory. This course aims to give a concise overview of the current state of the plasticity theory, and then to show the fascinating possibility of this promising branch of solid mechanics. Many applications in mechanics, material science and technology require a comprehensive understanding and reliable representation of the elastoplastic behavior observed in a large class of engineering materials. In the last few decades several phenomenological theories have been developed on the macroscopic level. In the course special attention is paid to extensions of these classical models at the microscale level. Extensions of these classic models taking into account the formation of microstructures and the microheterogeneity of multiphase materials have attracted a more pronounced scientific interest rather recently. The role of microstructures becomes more and more important with a decreasing size of the considered material specimen because then scale-effects play a dominant role. Microstructure is indeed crucial, since plastic behavior typically results from the interaction of complex substructures on several length scales. The macroscopic behavior is then determined by appropriate averages over the (evolving) microstructure

Subjects covered by the lectures are

- The fundamentals of the mathematical theory of plasticity
- Basic Laws of Plasticity
- Perfect plasticity
- The Prandtl-Reuss relations
- The Levy-Mises relations
- Problems In Plane Stress
- Problems of the perfect elastoplasticity
- Analytic solutions of some problems in elastoplasticity
- Elastoplastic torsion of a circular shaft
- Tube subjected to combined torsion and simple traction
- Cyclic torsion of a composite with unidirectional fibres
- Problems with hardening
- Asymptotic behaviour: shakedown
- The Melan-Koiter Theorem
- Computational Plasticity

Learning outcomes of the course :

Through a deep understanding of the theory and the realization of a project, the student will be able to apply theoretical, asymptotical and numerical tools to solve plasticity theory problems. In particular:

- He will have a deep understanding of plasticity theories and will be able to summary, compare and explain them.
- He will have a deep understanding of the resolution methods of elastoplastic problems, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to apply the resolution methods to classical problems of plasticity theory.
- He will be able to analyse and to evaluate (justify and criticise) these methods.
- He will be able to analyse new problems.

Prerequisites and co-requisites/ Recommended optional programme components :

Basic knowledge in

- Solid mechanics
- Elasticity Theory
- Vector calculus
- Differential equations
- Partial differential Equations

Planned learning activities and teaching methods :

Exercises with professor assistance and personal project.

Mode of delivery (face-to-face ; distance-learning) :

Face-to-Face

Required readings :

- Kachanov L.M. Fundamentals of the Theory of Plasticiyt. Dover Publications, 2004. 496 p.
- Hoffman O., Sachs G. Introduction to The Theory of Plasticity for Engineers. Literary Licensing, LLC, 2012. 290 p. ISBN 978-1258431471.
- Han W., Reddy B.D. Plasticity: Mathematical Theory and Numerical Analysis (Interdisciplinary Applied Mathematics). Springer, 2013. 435 p.
- Negahban M. The mechanical and Thermodynamical Theory of Plasticity. CRC Press, 2012. 784 p.
- Borja R.I. Plasticity: Modeling and Computations. Springer, 2013. 300 p.
- Foundations of the Theory of Elasticity, Plasticity and Viscoelasticity. Apple Academic Press, 2012 p. 320 p.
- Gerard A. Maugin The Thermomecahnics of plasticity and Fracture. Cambridge: Cambridge University Press, 1992. 350 p.
- Onate E., Owen R. Computational Plasticity. Dordrecht: Springer, 2007. 270 p.
- Dunne F., Petrinic N. Introduction to Computational Plasticity. Oxford: Oxford University Press, 2005. 259 p.
- Chakrbarty J. Applied Plasticity. Dordrecht: Springer, 2010. 771 p.
- Hill R. Mathematical Theory of Plasticity. Oxford: Oxford University Press, 1998. 356 p.
- Lubliner J. Plasticity Theory. University of California at Berkeley, 2006. 540.
- Rusinko A., Rusinko K. Plasticity and Creep of Metals. Berlin: Springer, 2011. 442 p.

Assessment methods and criteria :

Evaluation is based on the realization of a project related to the use / development of the analytical and numeric methods specific to the plasticity theory and on an examination.
The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked.
Participation to the examination and achievement of the project are mandatory.

PLASTICITY THEORY

Course Contents

ELASTOPLASTIC CONSTITUTIVE FRAMEWORK

6.11.2. Local Residual Increment of Deformation Gradient
6.12. Plastic Increment of Macroscopic Piola–Kirchhoff Stress
6.13. Plastic Increment of Macroscopic Lagrangian Strain
6.14. Macroscopic Increment of Plastic Work
6.15. Nontransmissibility of Basic Crystal Inequality
6.16. Analysis of Second-Order Work Quantities
6.17. General Analysis of Macroscopic Plastic Potentials
6.17.1. Deformation Space Formulation
6.17.2. Stress Space Formulation
6.18. Transmissibility of Ilyushin's Postulate
6.19. Aggregate Minimum Shear and Maxi mum Work Principle
6.20. Macroscopic Flow Potential for Rate-Dependent Plasticity

POLYCRYSTALLINE MODELS
7.1. Taylor-Bishop-Hill Analysis
7.1.1. Polycrystalline Axial Stress-Strain Curve
7.1.2. Stresses in Grain
7.1.3. Calculation of Polycrystalline Yield Surface
7.2. Eshelby's Inclusion Problem of Linear Elasticity
7.2.1. Inclusion Problem
7.2.2. Inhomogeneity Problem
7.3. Inclusion Problem for Incrementally Linear Material
7.3.1. Dual Formulation
7.3.2. Analysis of Concentration Tensors
7.3.3. Finite Deformation Formulation
7.4. Self-Consistent Method
7.4.1. Polarization Tensors
7.4.2. Alternative Expressions for Polycrystalline Moduli
7.4.3. NonalignedCrystal s
7.4.4. Polycrystalline Pseudomoduli
7.5. Isotropic Aggregates of Cubic Crystals
7.5.1. Voigt andReus s Estimates
7.6. Elastoplastic Crystal Embedded in Elastic Matrix
7.6.1. Concentration Tensor
7.6.2. Dual-Concentration Tensor
7.6.3. Locally Smooth Yield Surface
7.6.4. Rigid-Plastic Crystal in Elastic Matrix
7.7. Elastoplastic Crystal Embedded in Elastoplastic Matrix
7.7.1. Locally Smooth Yield Surface
7.7.2. Rigid-Plastic Crystal in Rigid-Plastic Matrix
7.8. Self-Consistent Determination of Elastoplastic Moduli
7.8.1. Kroner-Budiansky-Wu Method
7.8.2. Hutchinson's Calculations
7.8.3. Berveiller and Zaoui Accommodation Function
7.8.4. Lin's Model
7.8.5. Rigid-Plastic Moduli
7.9. Development of Crystallographic Texture
7.10. Grain Size Effects

Literature

1. Chakrbarty J. Applied Plasticity. Dordrecht: Springer, 2010. 771 p.
2. Hill R. Mathematical Theory of Plasticity. Oxford: Oxford University Press, 1998. 356 p.
3. Gerard A. Maugin The Thermomecahnics of plasticity and Fracture. Cambridge: Cambridge University Press, 1992. 350 p.

4. Onate E., Owen R. Computational Plasticity. Dordrecht: Springer, 2007. 270 p.
5. Dunne F., Petrinic N. Introduction to Computational Plasticity. Oxford: Oxford University Press, 2005. 259 p.
6. Lubliner J. Plasticity Theory. University of California at Berkeley, 2006. 540.
7. Kachanov L.M. Foundations of Theory of Plasticity. Amsterdam: North-Holland, 1971.
8. Sokolnikoff I. S. Mathematical Theory of Elasticity. New York: McGraw- Hill, 1956.
9. Bridgman P. W. Studies in Large Plastic Flow and Fracture. New York: McGraw-Hill, 1952.
10. Lubarda V.A. Elastoplasticity Theory. Boca Raton, London, New York, Washington: CRC Press LLC, 2002. 627 p.

Problems (From Chakrbarty J. Applied Plasticity. Dordrecht: Springer, 2010. 771 p.)

Problems

1.1 For a certain application involving an elastic/plastic material, the stress–strain curve in the plastic range needs to be replaced by a straight line defined by $\sigma = Y + T\varepsilon$. The actual strain-hardening curve can be represented by $\sigma = Y(E\varepsilon/Y)^{n}$. If the linear strain-hardening law predicts the same area under the stress–strain curve as that given by the power law curve, over the range $0 \le \varepsilon \le \varepsilon_0$, when both the hardening laws are extended backward to $\varepsilon = 0$, show that

$$\frac{E\varepsilon_0}{Y} = \left(\frac{1+n}{1-n}\right)^{1/n}, \quad \frac{T}{E} = \frac{2n}{i-n}\left(\frac{1-n}{1+n}\right)^{1/n}.$$

1.2 For an element of work-hardening material yielding according to the von Mises yield criterion under biaxial compression, show that the principal stresses can be expressed in terms of the polar angle θ of the deviatoric stress vector as

$$\sigma_x = -\frac{2\bar{\sigma}}{\sqrt{3}}\cos\theta, \quad \sigma_y = -\frac{2\bar{\sigma}}{\sqrt{3}}\sin\left(\frac{\pi}{6}-\theta\right),$$

where $\bar{\sigma}$ is the equivalent stress. Show also that the components of the plastic strain increment, according to the Prandtl–Reuss flow rule, can be expressed in terms of the angle θ and the current plastic modulus H as

$$d\varepsilon_x^p = -\cos\left(\frac{\pi}{6}+\theta\right)\left(\frac{d\bar{\sigma}}{H}\right), \quad d\varepsilon_y^p = \sin\theta\left(\frac{d\bar{\sigma}}{H}\right).$$

1.3 For an element of von Miss material deforming under a plane a strain tension in the x-direction and a stress-free state in the y-direction, show that the applied stress and the deviatoric angle at the initial yielding are given by $\sigma_e = Y/\sqrt{c}$ and $2\cos\theta_e = \sqrt{3/c}$, where $c = 1 - \nu + \nu^2$. If a prismatic beam made of such a material having a depth 2 h is bent to an elastic/plastic curvature, so that the depth of the elastic core becomes 2c, prove that the bending couple M is given by

$$\frac{M}{M_e} = \frac{a^2}{h^2} + 2\sqrt{3}c\int_{c/h}^{1}(\cos\theta)\,\xi d\xi, \quad \xi = \frac{y}{h},$$

where M_e is the bending moment at the elastic limit. Assuming a mean value of $\cos\theta$, equal to $\sqrt{\cos\theta_e}$, obtain the moment–curvature relationship in the dimensionless form

$$\frac{M}{M_e} = m - (m-1)\left(\frac{\kappa_e}{\kappa}\right)^2, \quad m = \sqrt{\frac{3}{2}}(3c)^{1/4}.$$

1.4 An ideally plastic bar of circular cross section is rendered completely plastic by the combined action of an axial force N and a twisting moment T. If the ratio of the rate of extension to the rate of twist at the yield point is denoted by $a\alpha/3$, show that the normal and shear stress distributions over the cross section of the bar are given by

$$\frac{\sigma}{Y} = \frac{\alpha}{\sqrt{\alpha^2 + 3r^2/a^2}}, \quad \frac{\tau}{Y} = \frac{r/a}{\sqrt{\alpha^2 + 3r^2/a^2}}.$$

1.5 A block of isotropic material is compressed in the x-direction by a pair of rigid smooth dies, while the deformation in the y-direction is completely suppressed by constraints. If the material strain hardens linearly with a constant tangent modulus T, show that the polar equation of the stress path in the deviatoric plane is given by

$$\frac{\bar{\sigma}}{Y} = \left\{ \frac{\sqrt{3}\sin\theta - (1-2\nu)(T/E)\cos\theta_e}{\sqrt{3}\sin\theta_e - (1-2\nu)(T/E)\cos\theta} \right\}^{\alpha T/E} \exp\left[(\theta_e - \theta)\frac{\beta T}{E} \right],$$

where $\bar{\sigma}$ is the equivalent stress, and θ is the deviatoric angle having a value θ_e at the initial yielding, while α and β are dimensionless parameters defined as

$$\alpha = \frac{3 + (1-2\nu)^2 (T/E)^2}{3 + (1-2\nu)^2 (T/E)} \quad \beta = \frac{\sqrt{3}(1-2\nu)(1-T/E)}{3 + (1-2\nu)^2 (T/E)}.$$

1.6 The plastic modulus of a certain kinematically hardening material varies with the equivalent plastic strain according to the relation

$$H = Kn \exp\left(-n\bar{\varepsilon}^p\right),$$

where K and n are material constants. A specimen of this material is first pulled in tension until the longitudinal stress is equal to σ_0 and is then subjected to a complete loading cycle which involves a plastic strain amplitude of amount $\varepsilon*$. Show that the longitudinal stress at the end of the loading cycle exceeds σ_0 by the amount

$$\Delta\sigma = -K\left[1 - \exp(-4n\varepsilon^*)\right] \exp\left(-n\varepsilon^*\right).$$

1.7 For a material that hardens according to the combined hardening rule, the isotropic and kinematic parts of the plastic modulus H are assumed to be in the ration $\beta/(1-\beta)$, where β is a constant. Assumie the plastic modulus to be given by

$$H = Kn \exp\left(-n\bar{\varepsilon}^p\right)$$

Considering a complete loading cycle of a specimen involving a constant strain amplitude of amount ε^*, following a stress equal to σ_0 applied by simple tension, show that the tensile stress at the end of the cycle exceeds σ_0 by the amount

$$\Delta\sigma = K\left[1 + \exp(-2n\varepsilon^*)\right]\left[(2\beta - 1) + \exp\left(-2n\varepsilon^*\right)\right] \exp\left(-n\varepsilon^*\right).$$

1.8 The plastic yielding and flow of a certain isotropic material can be predicted with sufficient accuracy by modifying the von Mises yield criterion in the form

$$J_2\left(1 - \frac{9J_3^2}{4J_2^3}\right)^{1/3} = k^2$$

An infinite plate of uniform initial thickness h_0, and containing a circular hole of initial radius a_0, is expanded by the application of an increasing radial pressure p. The material obeys Tresca's yield criterion and its associated flow rule, and strain-hardens according to the law $\sigma = Y \exp(n\varepsilon)$, where n is a constant. Show that the current thickness h in the deformed plate varies with the initial radial distance r_0 according to the formula

$$\frac{h_0}{h} = \frac{1}{2+n}\left\{\left(\frac{\rho}{r_0}\right)^{n/(1+n)} + (1+n)\left(\frac{r_0}{\rho}\right)^{2/(1+n)}\right\}, \quad a \leq r \leq \rho$$

where ρ is the radius to the rigid/plastic boundary. Denoting the current radius of the hole by a, derive the expression

$$\frac{a}{\rho} = \frac{1}{2+n}\left\{1 + (1+n)\left(\frac{a_0}{\rho}\right)^{(2+n)/(1+n)}\right\}, \quad \frac{1}{2} \leq \frac{a}{\rho} \leq 1$$

In the hydrostatic bulging of circular diaphragm using a die aperture of radius a, let the shape of the bulge at each stage be assumed as a spherical cap of angular span equal to 2α. Show that this assumption is consistent with Tresca's associated flow rule for an arbitrary strain-hardening characteristic of the material when the thickness of the bulge is uniform at each stage. Obtain the associated strain distribution in the form

$$\varepsilon_\theta = \frac{1}{2}\ln\left(\frac{1+\cos\phi}{1+\cos\alpha}\right) \quad \varepsilon_\phi = \frac{1}{2}\ln\left\{\left(\frac{2}{1+\cos\alpha}\right)\left(\frac{2}{1+\cos\phi}\right)\right\}$$

Consider a thin plate in the form of a ring sector defined by concentric circular arcs of radii a and b, where $b > a$. The plate is brought to the yield point state by the application of terminal couples that tend to increase the curvature. The radial and hoop stresses may be expressed in the form

$$\sigma_r = 2k\sin\alpha. \quad \sigma_\theta = \pm 2k\cos\left(\frac{\pi}{6} \mp \alpha\right)$$

which satisfies the von Mises yield criterion parametrically through α. The upper sign applies to $r \le c$, and the lower sign to $r \ge c$, where c is the radius to the neutral surface which corresponds to $\alpha = \alpha_0$. Using the equilibrium equation, show that

$$\frac{b^2}{c^2} = \frac{2}{\sqrt{3}}\cos\left(\frac{\pi}{6} - \alpha_0\right)\exp\left(\sqrt{3}\alpha_0\right), \quad \frac{c^2}{ab} = \left(1 - \frac{4}{3}\sin^2\alpha_0\right)^{-1/2}$$

For usual values of the b/a ratio in the preceding problem, the auxiliary angle α is fairly small, so that α^2 is generally negligible compared to unity. Show that the value of α at $r = c$, and the associated yield couple M_0, is given approximately by

$$\alpha_0 = \frac{\sqrt{3}}{4}\ln\left(\frac{b}{a}\right), \quad \frac{c^2}{ab} \approx 1 + \frac{1}{8}\left(\ln\frac{b}{a}\right)^2, \quad M_0 \equiv \frac{\sqrt{3}}{4}kh(b-a)^2$$

The last expression may be obtained by taking the hoop stress distribution approximately as

$$\sigma_\theta = k\left(\sqrt{3} + \alpha\right), \quad a \le r \le c; \quad \sigma_\theta = -k\left(\sqrt{3} - \alpha\right), \quad c \le r \le b$$

where α in the two regions is expressed in the way similar to that for α_0 .

Exercises

1 A polycrystalline metal has a plastic stress--strain curve that obeys Hollomon's equation, $\sigma = K$ εn.
Determine n, knowing that the flow stresses of this material at 2% and 10% plastic deformation (offset) are equal to 175 and 185 MPa, respectively.

2 You are traveling in an airplane. The engineer who designed it is, casually, on your side. He tells you that the wings were designed using the von Mises criterion. Would you feel safer if he had told you that the Tresca criterion had been used? Why?

3 A material is under a state of stress such that $\sigma 1 = 3\sigma 2 = 2\sigma 3$. It starts to flow when $\sigma 2 = 140$ MPa.
(a) What is the flow stress in uniaxial tension?
(b) If the material is used under conditions in which $\sigma 1 = -\sigma 3$ and $\sigma 2 = 0$, at which value of $\sigma 3$ will it flow, according to the Tresca and von Mises criteria?

4 A steel with a yield stress of 300 MPa is tested under a state of stress where $\sigma 2 = \sigma 1/2$ and $\sigma 3 = 0$. What is the stress at which yielding occurs if it is assumed that:
(a) The maximum-normal-stress criterion holds?
(b) The maximum-shear-stress criterion holds?
(c) The distortion-energy criterion holds?

5 Determine the maximum pressure that a cylindrical gas reservoir can withstand, using the three flow criteria. Use the following information: Material: AISI 304 stainless steel --- hot finished and annealed, $\sigma 0 = 205$ MPa Thickness: 25 mm
Diameter: 500 mm

Length: 1 mm
Hint: Determine the longitudinal and circumferential (hoop) stresses by the method of sections.
6 Determine the value of Poisson's ratio for an isotropic cube being plastically compressed between two parallel plates.
7 A low-carbon-steel cylinder, having a height of 50 mm and a diameter of 100 mm, is forged (upset) at 1,200◦C and a velocity of 1 m/s, until its height is equal to 15 mm. Assuming an efficiency of 60%, and assuming that the flow stress at the specified strain rate is 80 MPa, determine the power required to
forge the specimen.
8 Obtain the work-hardening exponent n using Considere's criterion for the curve of Example 3.4.
9 The stress--strain curve of a 70--30 brass is described by the equation $\sigma = 600\varepsilon 0.35\ p$ MPa until the onset of plastic instability.
(a) Find the 0.2% offset yield stress.
(b) Applying Considиre's criterion, find the real and engineering stress at the
onset of necking.
10 The onset of plastic flow in an annealed AISI 1018 steel specimen is marked by a load drop and the formation of a LËuders band. The initial strain rate is 10−4 s−1, the length of the specimen is 5 cm, and the LËuders plateau extends itself for a strain equal to 0.1. Knowing that each LËuders band is capable of producing a strain of 0.02 after its full motion, determine:
(a) The number of LËuders bands that traverse the specimen.
(b) The velocity of each LËuders band, assuming that only one band exists at each time.
11 A tensile test on a steel specimen having a cross-sectional area of 2 cm2 and length of 10 cm is conducted in an Instron universal testing machine with stiffness of 20 MN/m. If the initial strain rate is 10−3 s−1, determine the
slope of the load-extension curve in the elastic range (E = 210 GN/m2).
12 You have a piece of steel, and you are able to measure its hardness: HV = 250 kg/mm2. What is its estimated yield stress, in MPa?
13 You received a piece of cast iron, and you want to estimate its yield strength. You are able tomake a hardness indentation using a 10mmdiameter tungsten carbide sphere. The diameter of the indentation is 4 mm. What is the estimated yield strength?
14 Describe the similarities and differences in the phenomena of LËuders band formation in low-carbon steels and tensile drawing of a polymer.
15 The shear yield strength of a polymer is 30% higher in compression than in tension. Determine the coefficient A that represents the dependence of yield stress on hydrostatic pressure.
16 Looking at Figure 3.3, give reasons as to why the ultimate tensile stress (UTS) of AISI 1040 steel decreses with increased heat treatment.
17 **(a)** Describe the changes that occur at a microstructural level when a thin semicrystalline polymer begins to neck.
(b) Why does the strength increase in the load direction? Does the necking region become more or less transparent if the material is made of a semitransparent material?
18 The following stresses were measured on a metal specimen:
$\sigma 11$ = 94 MPa
$\sigma 22$ = 155 MPa
$\sigma 12$ = 85 MPa.
Determine the yielding for both the Tresca and von Mises criteria, given that
$\sigma 0$ = 180 MPa (yield stress). Which criterion is more conservative?
19 A flat indenter strikes the surface of an iron block and sinks into the material by 0.4 cm. Assuming that the surface of a piece of iron ($\tau\ 0$ = 6.6 GPa, $\sigma 0$ = 12.6 GPa, A = 0.5 cm2) can be modeled as triangular blocks, determine the force with which the indenter hits the material.
20 Determine the hardness of the copper specimen from the nanoindentation SEM image in Figure 3.42(b) knowing that the applied load is 2000 µN.
21 Calculate the projected area of an indentation made in keratin, the penetration depth h is 600 nm. Assume we used the Berkovich tip (A = 24.5h2).

22 You are designing a kinetic energy penetrator for the M1 tank. This penetrator is made of depleted (non-radioactive but highly lethal!) uranium- 0.75%Ti. Plot the stress--strain curve, from 0 to 1:
(**a**) At the following strain rates: 10−3 s−1, 103 s−1 (ambient temperature).
(**b**) At a strain rate of 10−3 s−1 and the following temperatures: 77 K, 100 K,
300 K. Given: Tm = 1473 K $\sigma 0$ = 1079 MPa K = 1120 MPa n = 0.25 C = 0.007 m = 1 $\dot{\varepsilon}$ = 10−4 s−1

Literature and Recommendations

1. Han W., Reddy B.D. Plasticity: Mathematical Theory and Numerical Analysis (Interdisciplinary Applied Mathematics). Springer, 2013. 435 p.

This book focuses on the theoretical aspects of small strain theory of elastoplasticity with hardening assumptions. It provides a comprehensive and unified treatment of the mathematical theory and numerical analysis. It is divided into three parts, with the first part providing a detailed introduction to plasticity, the second part covering the mathematical analysis of the elasticity problem, and the third part devoted to error analysis of various semi-discrete and fully discrete approximations for variational formulations of the elastoplasticity. This revised and expanded edition includes material on single-crystal and strain-gradient plasticity. In addition, the entire book has been revised to make it more accessible to readers who are actively involved in computations but less so in numerical analysis. The authors have written an excellent book which can be recommended for specialists in plasticity who wish to know more about the mathematical theory, as well as those with a background in the mathematical sciences who seek a self-contained account of the mechanics and mathematics of plasticity theory. In summary, the book represents an impressive comprehensive overview of the mathematical approach to the theory and numerics of plasticity. Scientists as well as lecturers and graduate students will find the book very useful as a reference for research or for preparing courses in this field. The book is professionally written and will be a useful reference to researchers and students interested in mathematical and numerical problems of plasticity. It represents a major contribution in the area of continuum mechanics and numerical analysis.

2. Foundations of The Theory of Elasticity, Plasticity and Viscoelasticity. Apple Academic Press, 2012 p. 320 p.

The book details fundamental and practical skills and approaches for carrying out research in the field of modern problems in the mechanics of deformed solids, which involves the theories of elasticity, plasticity, and viscoelasticity. The book includes all modern methods of research as well as the results of the authors' recent work and is presented with sufficient mathematical strictness and proof. The first six chapters are devoted to the foundations of the theory of elasticity. Theory of stress-strain state, physical relations and problem statements, variation principles, contact and 2D problems, and the theory of plates are presented, and the theories are accompanied by examples of solving typical problems. The last six chapters will be useful to postgraduates and scientists engaged in nonlinear mechanics of deformed inhomogeneous bodies. The foundations of the modern theory of plasticity (general, small elastoplastic deformations and the theory of flow), linear, and nonlinear viscoelasticity are set forth. Corresponding research of three-layered circular plates of various materials is included to illustrate methods of problem solving. Analytical solutions and numerical results for elastic, elastoplastic, lineaer viscoelastic and viscoelastoplastic plates are also given. Thermoviscoelastoplastic characteristics of certain materials needed for numerical account are presented in the eleventh chapter.

3. Negahban M. The mechanical and Thermodynamical Theory of Plasticity. CRC Press, 2012. 784 p.

The Mechanical and Thermodynamical Theory of Plasticity represents one of the most extensive and in-depth treatises on the mechanical and thermodynamical aspects of plastic and visicoplastic flow. Suitable for student readers and experts alike, it offers a clear and comprehensive presentation of multi-dimensional continuum thermodynamics to both aid in initial understanding and introduce and explore advanced topics. Covering a wide range of foundational subjects and presenting unique insights into the unification of disparate theories and practices, this book offers an extensive number of problems, figures, and examples to help the reader grasp the subject from many levels. Starting from one-dimensional axial motion in bars, the book builds a clear understanding of mechanics and continuum thermodynamics during plastic flow. This approach makes it accessible and applicable for a varied audience, including students and experts from engineering mechanics, mechanical engineering, civil engineering, and materials science.

4. Borja R.I. Plasticity: Modeling and Computations. Springer, 2013. 300 p.

“Plasticity Modeling & Computation” is a textbook written specifically for students who want to learn the theoretical, mathematical, and computational aspects of inelastic deformation in solids. It adopts a simple narrative style that is not mathematically overbearing, and has been written to emulate a professor giving a lecture on this subject inside a classroom. Each section is written to provide a balance between the relevant equations and the explanations behind them. Where relevant, sections end with one or more exercises designed to reinforce the understanding of the “lecture.” Color figures enhance the presentation and make the book very pleasant to read. For professors planning to use this textbook for their classes, the contents are sufficient for Parts A and B that can be taught in sequence over a period of two semesters or quarters.

5. Ibrahimbegovic A. Nonlinear Solid Mechanics. Theoretical Formulations and Finite Element Solution Methods. New York, Springer, 2009. 588 p.

The book was first and foremost written for students, engineers and expert users of different software products in nonlinear computational solid mechanics, as well as for those in teaching and research seeking to further enhance their understanding of the theoretical formulations which are the basis for development of the solution methods employed in computer codes. What is presented in this book is roughly the current state of the art of the finite element modelling of materials and structures, with the main difficulties (and their solutions) stemming from different industrial applications in mechanical, aerospace or civil engineering and material science.