Similarity Solutions of the Nonlinear Partial Differential Equations and Mechanics

Course contents:

The course is devoted to the similarity solutions of nonlinear problems arising in mechanics. Nonlinear problems have always tantalized scientists and engineers: they fascinate, but oftentimes elude exact treatment. A great majority of nonlinear problems are described by systems of nonlinear partial differential equations (PDEs) together with appropriate initial/boundary conditions; these model some physical phenomena. In the course it is shown that the nonlinear PDE systems with appropriate initial/ boundary conditions can now be solved effectively by means of the similarity solutions (exact solutions). The search for exact solutions is now motivated by the desire to understand the mathematical structure of the solutions and a deeper understanding of the physical phenomena described by them.
The course gives a modern presentation of dimensional analysis and the theory of dynamical similarity with numerous examples of various degree of complexity. New approaches of dimensional analysis and self-similarity hypotheses of new types are presented and applied to the investigation of various problems (creep crack problems, turbulent flows in atmospheric surface layers, wall layers of turbulent shear flows). The course will be of interest to students who wish to know the up-to-date methods of obtaining exact solutions to nonlinear problems.

Subjects covered by the lectures are

- Dimensional Analysis
- Similarity Variable and Similarity Presentation of Solutions
- Exact Travelling Wave Solutions
- Similarity Solutions of the First Kind
- Similarity Solutions of the Second Kind
- Intermediate Asymptotics
- Lie Group Theory
- Conservation Laws
- The Direct Method of Clarkson and Kruskal

Learning outcomes of the course :

Through a deep understanding of the theory and the realization of a project, the student will be able to find exact solutions of nonlinear problems by the use of different approaches (dimensional analysis, the Lie group theory, the direct method). In particular:

- He will have a deep understanding of creep theories and will be able to summary, compare and explain them.
- He will have a deep understanding of methods of obtaining the similarity solutions of nonlinear problems, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to apply the resolution methods to classical nonlinear problems of mechanics.
- He will be able to analyze and to evaluate (justify and criticise) these methods.
- He will be able to analyze new problems.

Prerequisites and co-requisites/ Recommended optional programme components :

Basic knowledge in

- Differential Equations

- Partial Differential Equations
- Fluid Mechanics
- Elasticity Theory
- Solid Mechanics

Planned learning activities and teaching methods :

Exercises with professor assistance and personal project.

Mode of delivery (face-to-face ; distance-learning) :

Face-to-Face

Required readings :

- Bluman G.W., Cole J.D. Similarity Methods for Differential Equations (Applied Mathematical Sciences. Vol 13.). Springer, 2013. 333 p. ISBN 978-038790177.
- Hornung H.G. Dimensional Analysis: Examples of the use of Symmetry. Dover publications, 2006. 80 p. ISBN 978-04864460059.
- Sachdev P.L. Self-Similarity and beyond. Exact Solutions of Nonlinear problems. Boca Raton London New York Washington: Charman and Hall/CRC, 2000. 315 p.
- Barenblatt G.I. Scaling Phenomena in Fluid Mechanics, 1994. 50 p.
- Barenblatt G. I. Scaling, Self-similarity, and Intermediate Asymptotics. New York: Cambridge University Press, 1996. 412 p. ISBN 978-0521435222.
- Barenblatt G. I. Similarity, Self-similarity, and Intermediate Asymptotics. New York: Consultants Bureau, 1979.
- Bluman G. W., Kumei S. Symmetries and Differential Equations. New York: Springer-Verlag, 1989.
- Kevorkian J., Cole, J. D. Multiple Scale and Singular Perturbation Methods. New York: Springer-Verlag,. 1996.
- Ovsiannikov L. V. Group Analysis of Differential Equations, Academic Press, New York, 1982.
- Sedov L.I. Similarity and Dimensional Methods in Mechanics. CRC Press, 1993. 496 p. ISBN 978-084939082.
- Hydon P.E. Symmetry Methods for Differential Equations. A Beginner's guide. Cambridge University Press, 200. 228 p.
- Miller W. Symmetry and Separation of Variables: Encyclopedia of Mathematics and its Applications: Volume 4. Cambridge University Press, 2012. ISBN 978-0-521-17739-9 .
- Giga M.N., Giga Y., Saal J. Nonlinear Partial Differential Equations. Asymptotic Behavior of Solutions and Self-Similar Solutions, Birkhauser, Boston, Basel New York: Springer, 2010. 307 p.

Assessment methods and criteria :

Evaluation is based on the realization of a project related to the use / development of analytical methods specific to similarity and on an examination.
The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked.
Participation to the examination and achievement of the project are mandatory.

Teaching Method: Class participation is mandatory. Everyone is expected to participate in discussions relating to reading materials, homework, exams and lectures.

Guaranteed Recipe for Success:
1) Take notes during lecture and sections.
2) After each lecture but before the next lecture review your notes. Identify the parts you do not understand.
3) Come to each lecture and discussion section with specific questions.
4) Keep up with the reading so that you have some familiarity with each topic prior to hearing about it in the lecture.
5) Find at least one "partner" in the class with whom you can meet at least once or twice a week to discuss materials from the lectures, the reading assignments and the homework.
6) Take the homework assignment seriously. Do not try to do the whole assignment the night before it is due. Some version of the homework questions will appear on the exams.

Course Contents
Similarity Solutions
1 Introduction. Dimensional analysis. Scaling. Examples. Very intense explosion. Modified problem of very explosion. Dimensional analysis. Intermediate Asymptotics. Renormalization group. Singular expansions. Intermediate asymptotics. Turbulent flow in pipes. Intermediate asymptotics. Fatigue crack propagation.
2 First-Order Partial Differential Equations
2.1 Linear Partial Differential Equations of First Order
2.2 Quasilinear Partial Differential Equations of First Order
2.3 Reduction of ut +unux +H(x, t, u) =0
2.4 Initial Value Problem for ut +g(u)ux +_h(u) =0
2.5 Initial Value Problem for ut +u_ux +_u_ =0
3 Exact Similarity Solutions of Nonlinear PDEs
3.1 Reduction of PDEs by Infinitesimal Transformations
3.2 Systems of Partial Differential Equations
3.3 Self-Similar Solutions of the Second Kind
3.4 A Nonlinear Heat Equation in Three Dimensions
3.5 Similarity Solution of Burgers Equation by the Direct Method
3.6 Exact Free Surface Flows for Shallow-Water Equations
3.7 An Example from Gasdynamics
4 Exact Travelling Wave Solutions
4.1 Travelling Wave Solutions
4.2 Simple Waves in 1-D Gasdynamics
4.3 Elementary Nonlinear Diffusive Travelling Waves
4.4 Travelling Waves for Higher-Order Diffusive Systems
4.5 Multidimensional Homogeneous Partial Differential Equations
4.6 Systems of Nonhomogeneous Partial Differential Equations
4.7 Exact Hydromagnetic Travelling Waves
4.8 Exact Simple Waves on Shear Flows
5 Exact Linearisation of Nonlinear PDEs
5.1 Introduction
5.2 Comments on the Solution of Linear PDEs
5.3 Burgers Equation in One and Higher Dimensions
5.4 Nonlinear Degenerate Diffusion Equation
5.5 Motion of Compressible Isentropic Gas in the Hodograph Plane
5.6 The Born-Infeld Equation
5.7 Water Waves up a Uniformly Sloping Beach
5.8 Simple Waves on Shear Flows
5.9 C-Integrable Nonlinear PDEs

6 Nonlinearisation and Embedding of Special Solutions
6.1 Introduction
6.2 Generalised Burgers Equations
6.3 Burgers Equation in Cylindrical Coordinates with Axisymmetry
6.4 Nonplanar Burgers Equation – A Composite Solution
6.5 Modified Burgers Equation
6.6 Embedding of Similarity Solution in a Larger Class
7 Asymptotic Solutions by Balancing Arguments
7.1 Asymptotic Solution by Balancing Arguments
7.2 Nonplanar Burgers Equation
7.3 One-Dimensional Contaminant Transport through Porous Media
8 Series Solutions of Nonlinear PDEs
8.1 Introduction
8.2 Analysis of Expansion of a Gas Sphere (Cylinder) into Vacuum
8.3 Collapse of a Spherical or Cylindrical Cavity
8.4 Converging Shock Wave from a Spherical or Cylindrical Piston

Further Reading

1. Polyanin A.D., Zaitsev V.F. Handbook of nonlinear partial differential equations. Boca Raton: Chapman and Hall/CRC, 2004. 814 p.
2. Barenblatt G.I. Scaling. Cambridge University Press, 2003.
3. Ibragimov N.H. CRC Handbookof Lie Group to Differential Equations. Vol.1. Symmetries, Exact Solutions and Conservation Laws. Boca Raton: CRC Press, 1994.
4. Ibragimov N.H. CRC Handbookof Lie Group to Differential Equations. Vol.2. Applications in Engineering and Physical Sciences. Boca Raton: CRC Press, 1995.
5. Bluman G.W., Cole J.D. Similarity Methods for Differential Equations. New York: Springer-Verlag, 1974.
6. Dresner L. Similarity Solutions of Nonlinear Partial Differential Equations. Boston: Pitman, 1983.

Exercises

1. Using the dimensional analysis derive a formula for the period of oscillation $T = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum, g is acceleration of gravity.
2. Consider the Blasius problem: the steady incompressible flow past a flat plate.
3. Find the similarity solutions of Burgers equations by the direct method:

$$u_t + uu_x + \frac{ju}{2t} = u_{xx}$$

$$u_t + u^2 u_x + \frac{ju}{2t} = u_{xx}$$

$$u_t + u^2 u_x = u_{xx}$$

$$u_t + uu_x + f(x,t) = g(t)u_{xx}$$

$$u_t + u^\beta u_x + f(t)u^\alpha = g(t)u_{xx}$$

4. Show that the following generalized Burgers equation have the given similarity solutions.

$$u_t + uu_x + \frac{ju}{2t} = u_{xx}, \quad j > 0$$
$$u(x,t) = \frac{x}{2t} + \frac{\alpha d_0}{b_0} t^{\alpha - 1/2} + b_0 t^{-1/2} U(z), where$$
$$z = -b_0 x t^{-1/2} + d_0 t^{\alpha}$$
$$U'' + UU' - \lambda U + az = 0.$$

5. Show that the following generalized Burgers equation have the given similarity solutions.

$$u_t + u^2 u_x + \frac{ju}{2t} = u_{xx}$$
$$u(x,t) = t^{-1/4} U(z), \quad where \quad z = xt^{-1/2}$$
$$U'' - U^2 U' + 2zU' + 2\left(\frac{1}{2} - j\right) U = 0.$$

6. Show that the following generalized Burgers equation have the given similarity solutions.

$$u_t + u^2 u_x = u_{xx}$$
$$u(x,t) = t^{-1/4} U(z), \quad where \quad z = xt^{-1/2}$$
$$U'' - U^2 U' + \frac{1}{2} zU' + \frac{1}{4} U = 0$$

7. Show that the following generalized Burgers equation have the given similarity solutions.

$$u_t + uu_x + f(x,t) = g(t) u_{xx}$$
$$u(x,t) = -xB'(t)/B(t) + K(t) + B(t)U(z)$$
$$z = -xB(t)/g(t) + D(t)$$
$$gD' + aB^2 D - KB = 0$$
$$U'' + UU' + azU' + F(z) = 0$$

Some nonlinear partial differential equations having the similarity solutions

Equation	Name of Equation	Similarity Solutions	
$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left[f(w) \frac{\partial w}{\partial x} \right]$	Nonstationary Heat Equation	$w = w(z)$, $z = xt^{-1/2}$	$[f(w)w']' + \frac{1}{2} zw' = 0$
$\frac{\partial w}{\partial t} = a \frac{\partial}{\partial x}\left(w^n \frac{\partial w}{\partial x} \right) + bw^k$	Nonstationary Heat Equation with Source	$w = t^p u(z)$, $z = xt^q$, $p = \frac{1}{1-k}$, $q = \frac{k-n-1}{2(1-k)}$	$a\left(u^n u'\right)' - qzu' + bu^k - pu = 0$
$\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + bw \frac{\partial w}{\partial x}$	Burgers Equation	$w = t^{-1/2} u(z)$, $z = xt^{-1/2}$	$au'' + buu' + \frac{1}{2} zu' + \frac{1}{2} u = 0$
$\frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x} \right)^2$	Burgers Equation	$w = w(z)$, $z = xt^{-1/2}$	$aw'' + b(w')^2 + \frac{1}{2} zw' = 0$

$$\frac{\partial w}{\partial t}=a\left(\frac{\partial w}{\partial x}\right)^k\frac{\partial^2 w}{\partial x^2}$$	Nonlinear Filtration Equation	$w=t^p u(z)$, $z=xt^q$ $p=-\frac{(k+2)q+1}{k}$, $q-arbitrary$	$a\left(u'\right)^k u''=$ $=qzu'+pu$
$$\frac{\partial w}{\partial t}=f\left(\frac{\partial w}{\partial x}\right)\frac{\partial^2 w}{\partial x^2}$$	Nonlinear Filtration Equation	$w=t^{1/2}u(z)$ $z=xt^{-1/2}$	$2f\left(u'\right)u''+zu'-$ $-u=0$
$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial x}\left[f(w)\frac{\partial w}{\partial x}\right]$$	Wave Equation	$w=w(z)$, $z=x/t$	$\left(z^2 w'\right)'=\left[f(w)w'\right]'$
$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=aw^n$$	Heat Equation with Source	$w=x^{\frac{2}{1-n}}u(z)$ $z=y/x$	$\left(1+z^2\right)u''-\frac{2(1+n)}{1-n}zu'+$ $+\frac{2(1+n)}{\left(1-n\right)^2}u-au^n=0$
$$\frac{\partial^2 w}{\partial x^2}+a\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}=0$$	Equation of Transonic Gas Flow	$w=x^{-3k-2}u(z)$ $z=x^k y$, $k\ is\ arbitrary$	$\frac{a}{k+1}u'u''+\frac{k^2}{k+1}z^2u''-$ $-5kzu'+3(3k+2)u=0$
$$\frac{\partial w}{\partial t}=a\frac{\partial^3 w}{\partial x^3}+bw\frac{\partial w}{\partial x}$$	Korteweg-de Vries Equation	$w=t^{-2/3}u(z)$ $z=xt^{-1/3}$	$au'''+buu'+$ $+\frac{1}{3}zu'+\frac{2}{3}u=0$
$$\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}=a\frac{\partial^3 w}{\partial y^3}$$	Boundary Layer Equation	$w=x^{\lambda+1}u(z)$ $z=x^\lambda y$ $\lambda\ is\ arbitrary$	$\left(2\lambda+1\right)\left(u'\right)^2-$ $\left(\lambda+1\right)uu''=au'''$

Literature and Recommendations

1. Barenblatt G.I. Scaling. Oxford: Cambridge University Press, 2003. 188 p.

Scaling (power-type) laws reveal the fundamental property of the phenomena-self similarity. Self-similar (scaling) phenomena repeat themselves in time and/or space. The property of self-similarity simplifies substantially the mathematical modeling of phenomena and its analysis-experimental, analytical and computational. The book begins from a non-traditional exposition of dimensional analysis, physical similarity theory and general theory of scaling phenomena. Classical examples of scaling phenomena are presented. It is demonstrated that scaling comes on a stage when the influence of fine details of initial and/or boundary conditions disappeared but the system is still far from ultimate equilibrium state (intermediate asymptotics). It is explained why the dimensional analysis as a rule is insufficient for establishing self-similarity and constructing scaling variables. Important examples of scaling phenomena for which the dimensional analysis is insufficient (self-similarities of the second kind) are presented and discussed. A close connection of intermediate asymptotics and self-similarities of the second kind with a fundamental concept of theoretical physics, the renormalization group, is explained and discussed. Numerous examples from various fields--from theoretical biology to fracture mechanics, turbulence, flame propagation, flow in porous strata, atmospheric and oceanic phenomena are presented for which the ideas of scaling, intermediate asymptotics, self-similarity and renormalization group were of decisive value in modeling.

2. Hornung H.G. Dimensional Analysis: Examples of the use of Symmetry. Dover publications, 2006. 80 p. ISBN 978-04864460059.

Derived from a course in fluid mechanics, this text for advanced undergraduates and beginning graduate students employs symmetry arguments to illustrate the principles of dimensional analysis. The examples provided demonstrate the effectiveness of symmetry arguments, and students will find these methods applicable to a wide field of interests.