Similarity Solutions of the Nonlinear Partial Differential Equations and Mechanics

Course contents:

The course is devoted to the similarity solutions of nonlinear problems arising in mechanics. Nonlinear problems have always tantalized scientists and engineers: they fascinate, but oftentimes elude exact treatment. A great majority of nonlinear problems are described by systems of nonlinear partial differential equations (PDEs) together with appropriate initial/boundary conditions; these model some physical phenomena. In the course it is shown that the nonlinear PDE systems with appropriate initial/boundary conditions can now be solved effectively by means of the similarity solutions (exact solutions). The search for exact solutions is now motivated by the desire to understand the mathematical structure of the solutions and a deeper understanding of the physical phenomena described by them.

The course gives a modern presentation of dimensional analysis and the theory of dynamical similarity with numerous examples of various degree of complexity. New approaches of dimensional analysis and self-similarity hypotheses of new types are presented and applied to the investigation of various problems (creep crack problems, turbulent flows in atmospheric surface layers, wall layers of turbulent shear flows). The course will be of interest to students who wish to know the up-to-date methods of obtaining exact solutions to nonlinear problems.

Subjects covered by the lectures are

- Dimensional Analysis
- Similarity Variable and Similarity Presentation of Solutions
- Exact Travelling Wave Solutions
- Similarity Solutions of the First Kind
- Similarity Solutions of the Second Kind
- Intermediate Asymptotics
- Lie Group Theory
- Conservation Laws
- The Direct Method of Clarkson and Kruskal

Learning outcomes of the course:

Through a deep understanding of the theory and the realization of a project, the student will be able to find exact solutions of nonlinear problems by the use of different approaches (dimensional analysis, the Lie group theory, the direct method). In particular:

- He will have a deep understanding of creep theories and will be able to summary, compare and explain them.
- He will have a deep understanding of methods of obtaining the similarity solutions of nonlinear problems, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to apply the resolution methods to classical nonlinear problems of mechanics.
- He will be able to analyze and to evaluate (justify and criticise) these methods.
- He will be able to analyze new problems.

Prerequisites and co-requisites/Recommended optional programme components:

Basic knowledge in

- Differential Equations
Planned learning activities and teaching methods:

Exercises with professor assistance and personal project.

Mode of delivery (face-to-face; distance-learning):

Face-to-Face

Required readings:


Assessment methods and criteria:

Evaluation is based on the realization of a project related to the use / development of analytical methods specific to similarity and on an examination.

The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked.

Participation to the examination and achievement of the project are mandatory.
**Teaching Method:** Class participation is mandatory. Everyone is expected to participate in discussions relating to reading materials, homework, exams and lectures.

**Guaranteed Recipe for Success:**
1) Take notes during lecture and sections.
2) After each lecture but before the next lecture review your notes. Identify the parts you do not understand.
3) Come to each lecture and discussion section with specific questions.
4) Keep up with the reading so that you have some familiarity with each topic prior to hearing about it in the lecture.
5) Find at least one "partner" in the class with whom you can meet at least once or twice a week to discuss materials from the lectures, the reading assignments and the homework.
6) Take the homework assignment seriously. Do not try to do the whole assignment the night before it is due. Some version of the homework questions will appear on the exams.

**Course Contents**

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2.3 Reduction of $ut +unux +H(x, t, u) =0$
2.4 Initial Value Problem for $ut +g(u)ux +_h(u) =0$
2.5 Initial Value Problem for $ut +u_ux +_u =0$
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7 Asymptotic Solutions by Balancing Arguments
7.1 Asymptotic Solution by Balancing Arguments
7.2 Nonplanar Burgers Equation
7.3 One-Dimensional Contaminant Transport through Porous Media
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8.3 Collapse of a Spherical or Cylindrical Cavity
8.4 Converging Shock Wave from a Spherical or Cylindrical Piston

Further Reading

Exercises
1. Using the dimensional analysis derive a formula for the period of oscillation

\[ T = 2\pi \sqrt{\frac{l}{g}} \], where \( l \) is the length of the pendulum, \( g \) is acceleration of gravity.

2. Consider the Blasius problem: the steady incompressible flow past a flat plate.
3. Find the similarity solutions of Burgers equations by the direct method:

\[ u_t + uu_x + \frac{ju}{2t} = u_{xx} \]
\[ u_t + u^2u_x + \frac{ju}{2t} = u_{xx} \]
\[ u_t + u^2u_x = u_{xx} \]
\[ u_t + uu_x + f(x,t) = g(t)u_{xx} \]
\[ u_t + u^\beta u_x + f(t)u^\alpha = g(t)u_{xx} \]
4. Show that the following generalized Burgers equation have the given similarity solutions.
5. Show that the following generalized Burgers equation have the given similarity solutions.

\[ u_t + uu_x + \frac{j u}{2t} = u_{xx}, \quad j > 0 \]
\[ u(x,t) = \frac{x}{2t} + \frac{\alpha d_0}{b_0} t^{\alpha - 1/2} + b_0 t^{-1/2} U(z), \text{where} \]
\[ z = -b_0 x t^{-1/2} + d_0 t^\alpha \]
\[ U'' + UU' - \lambda U + az = 0. \]

6. Show that the following generalized Burgers equation have the given similarity solutions.

\[ u_t + u^2 u_x + \frac{j u}{2t} = u_{xx} \]
\[ u(x,t) = t^{-1/4} U(z), \quad \text{where} \quad z = xt^{-1/2} \]
\[ U'' - U^2 U' + 2z U' + 2 \left( \frac{1}{2} - j \right) U = 0. \]

7. Show that the following generalized Burgers equation have the given similarity solutions.

\[ u_t + uu_x + f(x,t) = g(t) u_{xx} \]
\[ u(x,t) = -xB'(t)/B(t) + K(t) + B(t)U(z) \]
\[ z = -xB(t)/g(t) + D(t) \]
\[ gD' + aB^2 D - KB = 0 \]
\[ U'' + UU' + azU' + F(z) = 0 \]

<table>
<thead>
<tr>
<th>Equation</th>
<th>Name of Equation</th>
<th>Similarity Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[ f(w) \frac{\partial w}{\partial x} \right] )</td>
<td>Nonstationary Heat Equation</td>
<td>( w = w(z), \quad z = xt^{-1/2} )</td>
</tr>
<tr>
<td>( \frac{\partial w}{\partial t} = a \frac{\partial}{\partial x} \left( w^n \frac{\partial w}{\partial x} \right) + bw^k )</td>
<td>Nonstationary Heat Equation with Source</td>
<td>( w = t^pu(z), \quad z = xt^q )</td>
</tr>
<tr>
<td>( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + bw \frac{\partial w}{\partial x} )</td>
<td>Burgers Equation</td>
<td>( w = t^{-1/2} u(z), \quad z = xt^{-1/2} )</td>
</tr>
<tr>
<td>( \frac{\partial w}{\partial t} = a \frac{\partial^2 w}{\partial x^2} + b \left( \frac{\partial w}{\partial x} \right)^2 )</td>
<td>Burgers Equation</td>
<td>( w = w(z), \quad z = xt^{-1/2} )</td>
</tr>
</tbody>
</table>
\[
\frac{\partial w}{\partial t} = a \left( \frac{\partial w}{\partial x} \right)^k \frac{\partial^2 w}{\partial x^2}
\]

**Nonlinear Filtration Equation**

\[
w = t^p u(z), \\
z = xt^q \\
p = -\frac{(k + 2)q + 1}{k} \\
q \text{ is arbitrary}
\]

\[
a \left( u' \right)^k u'' = qzu' + pu
\]

\[
\frac{\partial w}{\partial t} = f \left( \frac{\partial w}{\partial x} \right) \frac{\partial^2 w}{\partial x^2}
\]

**Nonlinear Filtration Equation**

\[
w = t^{1/2} u(z), \\
z = xt^{-1/2}
\]

\[
2f \left( u' \right) u'' + zu' - u = 0
\]

\[
\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x} \left[ f(w) \frac{\partial w}{\partial x} \right]
\]

**Wave Equation**

\[
w = w(z), \\
z = x/t
\]

\[
(z^2 w')' = [f(w)w']'
\]

\[
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = aw^n
\]

**Heat Equation with Source**

\[
w = x^{1-n} u(z), \\
z = y/x
\]

\[
(1 + z^2) u'' - \frac{2(1+n)}{1-n} zu' + \frac{2(1+n)}{(1-n)2} u - au^n = 0
\]

\[
\frac{\partial^2 w}{\partial x^2} + a \frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial y^2} = 0
\]

**Equation of Transonic Gas Flow**

\[
w = x^{-3k-2} u(z), \\
z = x^k y, \\
k \text{ is arbitrary}
\]

\[
a \left( u' \right)^k u'' + \frac{k^2}{k+1} z^2 u'' - \frac{k+1}{5kzu' + 3(3k+2)u} = 0
\]

\[
\frac{\partial w}{\partial t} = a \frac{\partial^3 w}{\partial x^3} + bw \frac{\partial w}{\partial x}
\]

**Korteweg-de Vries Equation**

\[
w = t^{-\frac{2}{3}} u(z), \\
z = xt^{-1/3}
\]

\[
a u''' + buu' + \frac{1}{3} zu' + \frac{2}{3} u = 0
\]

\[
\frac{\partial w}{\partial y} \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} = a \frac{\partial^3 w}{\partial y^3}
\]

**Boundary Layer Equation**

\[
w = x^{2+l} u(z), \\
z = x^\lambda y, \\
\lambda \text{ is arbitrary}
\]

\[
\left( 2 \lambda + 1 \right) \left( u' \right)^2 - \left( \lambda + 1 \right) uu'' = au''
\]

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**Literature and Recommendations**


Scaling (power-type) laws reveal the fundamental property of the phenomena—self-similarity. Self-similar (scaling) phenomena repeat themselves in time and/or space. The property of self-similarity simplifies substantially the mathematical modeling of phenomena and its analysis-experimental, analytical and computational. The book begins from a non-traditional exposition of dimensional analysis, physical similarity theory and general theory of scaling phenomena. Classical examples of scaling phenomena are presented. It is demonstrated that scaling comes on a stage when the influence of fine details of initial and/or boundary conditions disappeared but the system is still far from ultimate equilibrium state (intermediate asymptotics). It is explained why the dimensional analysis as a rule is insufficient for establishing self-similarity and constructing scaling variables. Important examples of scaling phenomena for which the dimensional analysis is insufficient (self-similarities of the second kind) are presented and discussed. A close connection of intermediate asymptotics and self-similarities of the second kind with a fundamental concept of theoretical physics, the renormalization group, is explained and discussed. Numerous examples from various fields—from theoretical biology to fracture mechanics, turbulence, flame propagation, flow in porous strata, atmospheric and oceanic phenomena are presented for which the ideas of scaling, intermediate asymptotics, self-similarity and renormalization group were of decisive value in modeling.

Derived from a course in fluid mechanics, this text for advanced undergraduates and beginning graduate students employs symmetry arguments to illustrate the principles of dimensional analysis. The examples provided demonstrate the effectiveness of symmetry arguments, and students will find these methods applicable to a wide field of interests.