# Applied Solid Mechanics

## Course contents:

The subject of Elasticity is concerned with the determination of the stresses and displacements in a body as a result of applied mechanical or thermal loads, for those cases in which the body reverts to its original state on the removal of the loads. The theory of elasticity comprises a consistent set of equations which uniquely describe the state of stress, strain, and displacement at each point within an elastic deformable body. Engineering approaches are often based on a strength of materials formulation with its various specialized derivatives such as the theories of rods, beams, plates, and shells. The distinguishing feature between the various alternative approaches and the theory of elasticity is the pointwise, as opposed to sectional description embodied in elasticity. The basic elements of the theory are equilibrium equations relating the stresses; kinematic equations relating the strains and displacements; constitutive equations relating the stresses and strains; boundary conditions relating to the physical domain; and uniqueness constraints relating to the applicability of the solution. In this chapter, the mathematical preliminaries for the presentation, such as vector algebra, integral theorems, indicial notation, and Cartesian tensors, are introduced.

**Subjects covered by the lectures are**

- Vector and Tensor Algebra. Vectors, tensors and transformation rules.
- Notation for stress and displacement.
- Strains and their relation to displacements; Tensile strain; Rotation and shear strain.
- Two-dimensional Problems.
- Plane strain and plane stress; Plane strain; SaintVenant's principle; Stress function formulation; The concept of a scalar stress function; Choice of a suitable form; The Airy stress function.
- Complex Variable Formulation.
- Three dimensional problems; Displacement function solutions; The strain potential; The Galerkin vector; The Papkovich-Neuber solution; Change of coordinate system; Completeness and uniqueness; Methods of partial integration; Body forces; Conservative body force fields; Nonconservative body force fields; The Boussinesq potentials.
- Variational methods; Strain energy; Strain energy density; Conservation of energy; Potential energy of the external forces; Theorem of minimum total potential energy; Approximate solutions — the Rayleigh-Ritz method; Castigliano's second theorem; Approximations using Castigliano's second theorem.
- Wedge problems. Williams' asymptotic method. Acceptable singularities. Eigenfunction expansion. Nature of the eigenvalues. The singular stress fields.
- Plane contact problems. Self-similarity. The Flamant Solution. Distributed normal tractions. Frictionless contact problems. The flat punch. The cylindrical punch (Hertz problem). Problems with two deformable bodies. Uncoupled problems.

## Learning outcomes of the course :

Through a deep understanding of the theory and the realization of a project, the student will be able to apply theoretical, asymptotical and numerical tools to solve solid mechanics problems. In particular:

- He will have a deep understanding of Solid Mechanics and will be able to summary, compare and explain them.
- He will have a deep understanding of the resolution methods of boundary value problems of Solid Mechanics, and will be able to summary, compare and explain them. He will also know their application range.
- He will be able to apply the resolution methods to classical problems of Solid Mechanics.
- He will be able to analyze and to evaluate (justify and criticise) these methods.
- He will be able to analyze new problems.

**Prerequisites and co-requisites/ Recommended optional programme components :**

Basic knowledge in

- Differential Equations
- Partial Differential Equations
- Tensor Analysis

**Planned learning activities and teaching methods :**

Exercises with professor assistance and personal project.

**Mode of delivery (face-to-face ; distance-learning) :**

Face-to-Face

**Required readings :**

- Gould P. L. Introduction to Linear Elasticity. Springer, 2013. 359 p.
- EhrlacherA., Markenscoff X. Duality, Symmetry and Symmetry Lost in Solid Mechanics. Paris: Presses des Ponts, 2011. 396 p.
- Barber J.R. Elasticity. Springer, 2010. 538 p.
- Ibrahimbegovic A. Nonlinear Solid Mechanics. Theoretical Formulations and Finite Element Solution Methods. New York, Springer, 2009. 588 p.
- Constantinescu A., Korsunsky A. Elasticity with Mathematica. An Introduction to Continuum Mechanics and Linear Elasticity. Cambridge University Press, 2012. 266 p.
- Temam R., Miranville A. Mathematical Modelling in Continuum Mechanics. Cambridge University Press, 2005.

**Assessment methods and criteria :**

Evaluation is based on the realization of a project related to the use / development of numerical methods specific to applied solid mechanics and on an examination.
The examination is based on the whole content of the class. Problems similar to the ones studied during the classes, and new problems will be part of the questions. Justification using the theoretical content is also asked.
Participation to the examination and achievement of the project are mandatory.

**Teaching Method:** Class participation is mandatory. Everyone is expected to participate in discussions relating to reading materials, homework, exams and lectures.

**Guaranteed Recipe for Success:**
1) Take notes during lecture and sections.
2) After each lecture but before the next lecture review your notes. Identify the parts you do not understand.
3) Come to each lecture and discussion section with specific questions.
4) Keep up with the reading so that you have some familiarity with each topic prior to hearing about it in the lecture.

5) Find at least one "partner" in the class with whom you can meet at least once or twice a week to discuss materials from the lectures, the reading assignments and the homework.
6) Take the homework assignment seriously. Do not try to do the whole assignment the night before it is due. Some version of the homework questions will appear on the exams.

**Course Contents**

**Applied Solid Mechanics**
**1 Modelling Solids**
1.1 Introduction. 1.2 Hooke's law. 1.3 Lagrangian and Eulerian coordinates. 1.4 Strain. 1.5 Stress. 1.6 Conservation of momentum. 1.7 Linear elasticity. 1.8 The incompressibility approximation. 1.9 Energy. 1.10 Boundary conditions and well-posedness.1.11 Coordinate systems. 1.11.1 Cartesian coordinates.1.11.2 Cylindrical polar coordinates. 1.11.3 Spherical polar coordinates
**2 Linear Elastostatics**
2.1 Introduction. 2.2 Linear displacements. 2.2.1 Isotropic expansion. 2.2.2 Simple shear.2.2.3 Uniaxial stretching. 2.2.4 Biaxial strain. 2.2.5 General linear displacement. 2.3 Antiplane strain. 2.4 Torsion. 2.5 Multiply-connected domains. 2.6 Plane strain. 2.6.1 Definition. 2.6.2 The Airy stress function. 2.6.3 Boundary conditions. 2.6.4 Plane strain in a disc. 2.6.5 Plane strain in an annulus. 2.6.6 Plane strain in a rectangle. 2.6.7 Plane strain in a semi-infinite strip. 2.6.8 Plane strain in a half-space. 2.6.9 Plane strain with a body force. 2.7 Compatibility. 2.8 Generalised stress functions. 2.8.1 General observations. 2.8.2 Plane strain revisited. 2.8.3 Plane stress. 2.8.4 Axisymmetric geometry. 2.8.5 The Galerkin representation. 2.8.6 Papkovich{Neuber potentials. 2.8.7 Maxwell and Morera potentials. 2.9 Singular solutions in elastostatics. 2.9.1 The delta function. 2.9.2 Point and line forces. 2.9.3 The Green's tensor. 2.9.4 Point incompatibility.

**3 Linear Elastodynamics**
3.1 Introduction. 3.2 Normal modes and plane waves. 3.2.1 Normal modes. 3.2.2 Waves in the frequency domain. 3.2.3 Scattering. 3.2.4 *P*-waves and *S*-waves. 3.2.5 Mode conversion in plane strain. 3.2.6 Love waves. 3.2.7 Rayleigh waves. 3.3 Dynamic stress functions. 3.4 Waves in cylinders and spheres. 3.4.1 Waves in a circular cylinder. 3.4.2 Waves in a sphere. 3.5 Initial-value problems. 3.5.1 Solutions in the time domain. 3.5.2 Fundamental solutions. 3.5.3 Characteristics. 3.6 Moving singularities.
**4 Approximate Theories**
4.1 Introduction.
4.2 Longitudinal displacement of a bar. 4.3 Transverse displacements of a string. 4.4 Transverse displacements of a beam. 4.4.1 Derivation of the beam equation. 4.4.2 Boundary conditions. 4.4.3 Compression of a beam. 4.4.4 Waves on a beam. 4.5 Linear rod theory. 4.6 Linear plate theory. 4.6.1 Derivation of the plate equation. 4.6.2 Boundary conditions. 4.6.3 Simple solutions of the plate equation. 4.6.4 An inverse plate problem. 4.6.5 More general in-plane stresses. 4.7 Von Karman plate theory. 4.7.1 Assumptions underlying the theory. 4.7.2 The strain components. 4.7.3 The Von Karman equations. 4.8 Weakly curved shell theory. 4.8.1 Strain in a weakly curved shell. 4.8.2 Linearised equations for a weakly curved shell. 4.8.3 Solutions for a thin shell. 4.9 Nonlinear beam theory. 4.9.1 Derivation of the model. 4.9.2 Example: detection of a diving board. 4.9.3 Weakly nonlinear theory and buckling. 4.10 Nonlinear rod theory. 4.11 Geometrically nonlinear wave propagation. 4.11.1 Nonlinearity and solitons. 4.11.2 Gravity-torsional waves. 4.11.3 Travelling waves on a beam. 4.11.4 Weakly nonlinear waves on a beam.

**5 Nonlinear Elasticity**
5.1 Introduction. 5.2 Stress and strain revisited. 5.2.1 Deformation and strain. 5.2.2 The Piola-Kirchhoff stress tensors. 5.2.3 The momentum equation. 5.2.4 Example: one-dimensional nonlinear elasticity. 5.3 The constitutive relation. 5.3.1 Polar decomposition. 5.3.2 Strain invariants. 5.3.3 Frame indifference and isotropy. 5.3.4 The energy equation. 5.3.5 Hyperelasticity. 5.3.6 Linear elasticity. 5.3.7 Incompressibility. 5.3.8 Examples of constitutive relations. 5.4 Examples. 5.4.1

Principal stresses and strains. 5.4.2 Biaxial loading of a square membrane. 5.4.3 Blowing up a balloon. 5.4.4 Cavitation.

**6 Asymptotic Analysis**

6.1 Introduction. 6.2 The linear plate equation. 6.2.1 Nondimensionalisation and scaling. 6.2.2 Dimensionless equations. 6.2.3 Leading-order equations. 6.3 Boundary conditions and St Venant's principle. 6.3.1 Boundary layer scalings. 6.3.2 Equations and boundary conditions. 6.3.3 Solvability conditions. 6.3.4 Asymptotic expansions. 6.4 The von Karman plate equations and weakly curved shells. 6.4.1 Background. 6.4.2 Scalings. 6.4.3 Leading-order equations. 6.4.4 Equations for a weakly curved shell.

6.5 The Euler{Bernoulli plate equations. 6.5.1 Dimensionless equations. 6.5.2 Asymptotic structure of the solution. 6.5.3 Leading-order equations. 6.5.4 Longitudinal stretching of a plate. 6.6 The linear rod equations. 6.7 Linear shell theory. 6.7.1 Geometry of the shell. 6.7.2 Dimensionless equations. 6.7.3 Leading-order equations.

**7 Fracture and Contact Problems**

7.1 Introduction. 7.2 Static Brittle Fracture. 7.2.1 Physical background. 7.2.2 Mode III cracks. 7.2.3 Mathematical methodologies for crack problems. 7.2.4 Mode II cracks. 7.2.5 Mode I cracks. 7.2.6 Mixed loading.7.2.7 Dynamic fracture. 7.3 Contact. 7.3.1 Contact of elastic strings. 7.3.2 Other thin solids. 7.3.3 Smooth contact in plane strain. 7.4 Conclusions.

**8 Plasticity**

8.1 Introduction. 8.2 Models for granular material. 8.2.1 Static behaviour. 8.2.2 Granular flow. 8.3 Dislocation theory. 8.4 Perfect plasticity theory for metals. 8.4.1 Torsion problems. 8.4.2 Plane strain and plane stress conditions. 8.4.3 Three-dimensional yield conditions. 8.5 Plastic flow .8.6 Simultaneous Elasticity and Plasticity.

**9 More General Theories**

9.1 Introduction. 9.2 Viscoelasticity. 9.2.1 Introduction. 9.2.2 Springs and dashpots. 9.2.3 Three-dimensional linear viscoelasticity. 9.2.4 Large-strain viscoelasticity. 9.3 Thermoelasticity. 9.4 Composite Materials and Homogenisation. 9.4.1 One-dimensional homogenization. 9.4.2 Two-dimensional homogenization. 9.4.3 Three-dimensional homogenization. 9.5 Poroelasticity. 9.6 Anisotropy.

**Further Reading**

1. Howell P., Kozyreff G., Ockendon J. Applied Solids Mechanics. Cambridge Universities Press.2008. 452 p.

2. EhrlacherA., Markenscoff X. Duality, Symmetry and Symmetry Lost in Solid Mechanics. Paris: Presses des Ponts, 2011. 396 p.

3. Gould P. L. Introduction to Linear Elasticity. Springer, 2013. 359 p.

4. Barber J.R. Elasticity, Springer, 2010. 538 p.

5. Krenk S., *Non-linear Modelling and Analysis of Solids and Structures*, Cambridge University Press, Cambridge UK, 2009.

Recommended Literature

**1) Constantinescu A., Korsunsky A. Elasticity with Mathematica. An Introduction to Continuum Mechanics and Linear Elasticity. Cambridge University Press, 2012. 266 p.**

This book introduces key ideas and principles in the theory of elasticity with the help of symbolic computation. Differential and integral operators on vector and tensor fields of displacements, strains and stresses are considered on a consistent and rigorous basis with respect to curvilinear orthogonal coordinate systems. As a consequence, vector and tensor objects can be manipulated readily, and fundamental concepts can be illustrated and problems solved with ease. The method is illustrated using a variety of plane and three-dimensional elastic problems. General theorems, fundamental solutions, displacements and stress potentials are presented and discussed. The Rayleigh-Ritz method for obtaining approximate solutions is introduced for elastostatic and spectral analysis problems. Containing more than 60 exercises and solutions in the form of Mathematica notebooks that accompany every chapter, the reader can learn and master the techniques while applying them to a large rang of practical and fundamental problems.

2) **Sciamarella C.A., Sciamarella F.M. Experimental Mechanics of Solids. Willey, 2012. 776 p.**

Experimental Mechanics of Solids is a comprehensive introduction to the topics, technologies and methods of experimental mechanics of solids. It begins by establishing the fundamentals of continuum mechanics, explaining key areas such as the equations used, stresses and strains, and two and three dimensional problems. Having laid down the foundations of the topic, the book then moves on to look at specific techniques and technologies with emphasis on the most recent developments such as optics and image processing. Most of the current computational methods, as well as practical ones, are included to ensure that the book provides information essential to the reader in practical or research applications. Key features of the book:

- Presents widely used and accepted methodologies that are based on research and development work of the lead author
- Systematically works through the topics and theories of experimental mechanics including detailed treatments of the Moire, Speckle and holographic optical methods
- Includes illustrations and diagrams to illuminate the topic clearly for the reader
- Provides a comprehensive introduction to the topic, and also acts as a quick reference guide

This comprehensive book forms an invaluable resource for graduate students and is also a point of reference for researchers and practitioners in structural and materials engineering.

**3) Temam R., Miranville A. Mathematical Modelling in Continuum Mechanics. Cambridge University Press, 2005.**

The book is a unique resource for those studying continuum mechanics at the advanced undergraduate and beginning graduate level whether in engineering, mathematics, physics, or the applied sciences. Exercises and hints for solutions have been added to the majority of chapters, and the final part on solid mechanics has been substantially expanded. These additions have now made it appropriate for use as a textbook, but it also remains an ideal reference book for students and anyone interested in continuum mechanics.